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Characterization of the Thermal-Wave Field in a Wedge-Shaped Solid Using the Green's Function Method

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Abstract In this study, a theoretical model is established for a wedge-like solid with an open sector surrounded by walls of radius R of a cylindrical rod illuminated by a modulated circular Gaussian incident beam by means of the Green's function method in cylindrical coordinates. An analytical expression for the thermal-wave field in such a sample is presented. The theory is validated by reducing the arbitrary geometrical structure of the wedge to simpler geometries. It is shown that the frequency dependence of the thermal-wave field near the edge exhibits a large phase lag compared with that at a location far from the edge. The theory provides a foundation for quantitatively characterizing wedge-shaped industrial samples, such as metals with sintered edges, using photothermal methods in a non-contact and non-destructive manner.

Keywords Green's function method · Thermal-wave fields · Wedge-like solids

1 Introduction

Laser-induced photothermal radiometry (PTR) has become a powerful tool for the thermophysical characterization and non-destructive evaluation (NDE) of broad classes of materials due to its non-destructive and highly sensitive nature. With increasing applications of PTR to the characterization of materials with a curved surface, studies

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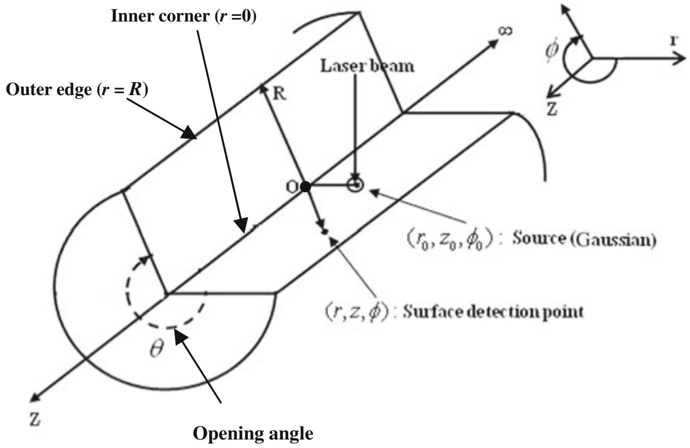

Fig. 1 Geometry and coordinates of a wedge-shaped solid

of the photothermal responses of these types of solids (e.g., cylindrical or spherical samples) have been reported in recent years [1–8]. In this study, we establish a theoretical model for a class of wedge-shaped structures with an open sector surrounded by walls of arbitrary radius R of a cylindrical rod, and illuminated with a modulated circular Gaussian laser beam. The study is based on a generalization of the thermal-wave Green's function method in cylindrical coordinates. Based on the theoretical model, the thermal-wave field of arbitrary wedge-shaped surfaces, i.e., wedges with arbitrary opening angles between 0° and 360°, and the frequency response of any point on the wall surfaces of the wedge can be obtained.

2 Theory

The geometry and coordinates of the considered wedge-shaped structure are shown in Fig. 1. The Green's function for the cylindrical sector of infinite height, radius R, and opening angle θ, can be obtained by assuming a spatially impulsive thermal-wave source located at (r_0, z_0, ϕ_0) and homogeneous Neumann conditions along all bounding surfaces. In cylindrical coordinates, the Green's function satisfies [9]

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}G(r\,|r_0\,;\omega)\right]+\frac{\partial^2}{\partial z^2}G(r\,|r_0\,;\omega)+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}G(r\,|r_0\,;\omega)-\sigma^2G(r\,|r_0\,;\omega)$$
$$=-\frac{1}{\alpha r}\delta(r-r_0)\delta(z-z_0)\delta(\phi-\phi_0) \quad (1)$$

Using separation of variables with $G(r, z, \phi\,|\vec{r}_0\,;\omega) = R(r)Z(z)\Phi(\phi)$, the boundary conditions represent an insulating thermal-wave flux (homogeneous Neumann) on the surface of the corner, located at $\phi = 0$ and θ. As a result of extensive algebraic calculations, the following Green's function is obtained:

$$
\begin{aligned}
&G(r, z, \phi\,|r_0, z_0, \phi_0;\quad \omega)\\
&= \frac{1}{\theta\alpha R^2}\sum_{m=1}^{\infty}\left\{\frac{J_0(\beta_m r)J_0(\beta_m r_0)\mathrm{e}^{-\xi_{0m}|z-z_0|}}{\xi_{0m}J_0^2(\beta_m R)}\right.\\
&\quad\left.+\,2\sum_{n=1}^{\infty}\frac{J_{n\pi/\theta}(\lambda_{nm}r)J_{n\pi/\theta}(\lambda_{nm}r_0)\mathrm{e}^{-\xi_{nm}|z-z_0|}\cos(\frac{n\pi\phi}{\theta})\cos(\frac{n\pi\phi_0}{\theta})}{\xi_{nm}[1-(n\pi/\theta\lambda_{nm}R)^2]J_{n\pi/\theta}^2(\lambda_{nm}R)}\right\},
\end{aligned}
\tag{2}
$$

where $\xi_{nm}(\omega) = (\lambda_{nm}^2 + \frac{\mathrm{i}\omega}{\alpha})^{\frac{1}{2}}$. $J_x\ (\cdot)$ represents a Bessel function of the first kind of fractional order x. λ_{nm} is the m-th root of the Bessel function $J_{n\pi/\theta}(\lambda R)$ which satisfies

$$
\mathrm{d}J_{n\pi/\theta}(\lambda r)/\mathrm{d}r\ |_{r=R}\ = 0 \tag{3}
$$

For a laser beam incident on the surface $\phi = 0°$, the thermal-wave field for an opaque material (no volume source) such as a metal wedge is given by [9]

$$
T(\vec{r},\omega) = \alpha\oiint_{S_0}[G(\vec{r}\,|\,\vec{r}_0^{\,s};\omega)\vec{\nabla}_0 T(\vec{r}_0^{\,s},\omega)]\cdot\mathrm{d}\vec{S}_0 \tag{4}
$$

Assuming an incident Gaussian laser beam centered at $(\rho_0, 0, 0)$, we can write the boundary condition

$$
k\hat{n}\cdot\vec{\nabla}T(r, z, 0; \omega) = \frac{1}{2}F_0\exp\left\{-\left[(r-\rho_0)^2 + z^2\right]/W^2\right\}\left(1+\mathrm{e}^{\mathrm{i}\omega t}\right), \tag{5}
$$

where F_0 is the optical flux at the surface $(\mathrm{W}\cdot\mathrm{m}^{-2})$. The thermal-wave field should be the same when the detection point is very close to the corner ($r \to 0$) with respect to the discontinuity at the corner $r = 0$:

$$
T(r, z, \phi; \omega)\,\big|_{\phi=0,r=0}\ = T(r, z, \phi; \omega)\,\big|_{\phi=\theta,r=0}\ . \tag{6}
$$

It is further assumed that the incident laser beam does not straddle the corner at $r = 0$. The final expression for the thermal-wave field is

$$
\begin{aligned}
&T(r, z, \phi; \omega)\\
&= \frac{F_0}{2\theta kR^2}\sum_{m=1}^{\infty}\left\{\frac{J_0(\beta_m r)}{\xi_{0m}J_0^2(\beta_m R)}\int_{-\infty}^{\infty}\mathrm{e}^{-\xi_{0m}|z-z_0|-(z_0/W)^2}\mathrm{d}z_0\int_0^R J_0(\beta_m r_0)\mathrm{e}^{-(r_0-\rho_0)^2/W^2}\mathrm{d}r_0\right.\\
&\quad+2\sum_{n=1}^{\infty}\frac{J_{n\pi/\theta}(\lambda_{nm}r)\cos(\frac{n\pi\phi}{\theta})}{\xi_{nm}[1-(n\pi/\theta\lambda_{nm}R)^2]J_{n\pi/\theta}^2(\lambda_{nm}R)}\int_{-\infty}^{\infty}\mathrm{e}^{-\xi_{nm}|z-z_0|-(z_0/W)^2}\mathrm{d}z_0\\
&\quad\left.\times\int_0^R J_{n\pi/\theta}(\lambda_{nm}r_0)\mathrm{e}^{-(r_0-\rho_0)^2/W^2}\mathrm{d}r_0\right\}.
\end{aligned}
\tag{7}
$$

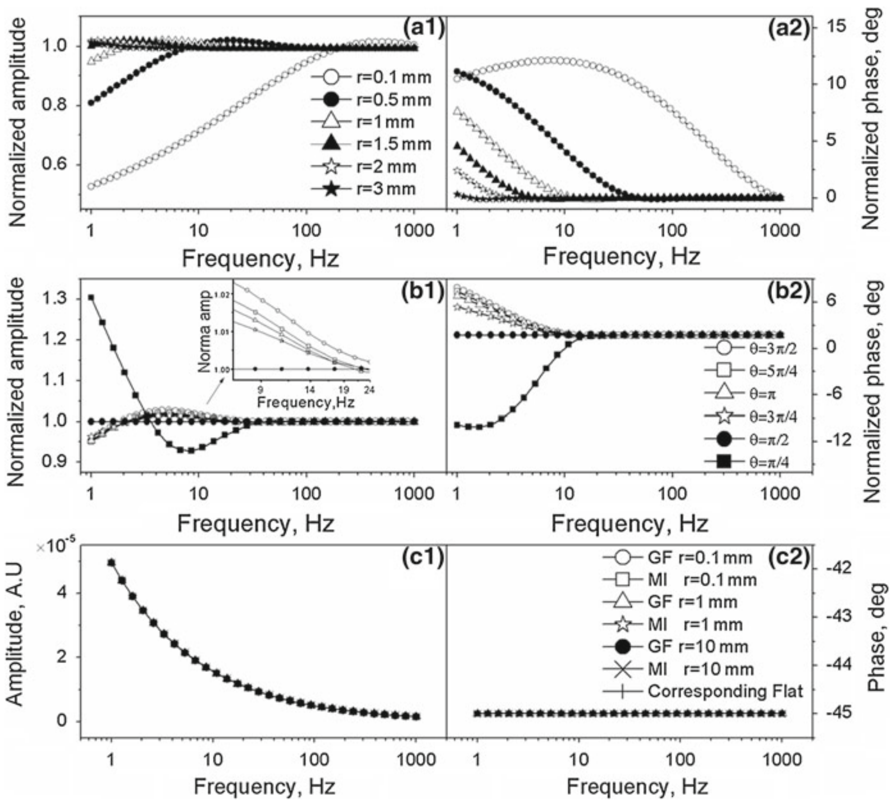

Fig. 2 (a1, a2) Thermal-wave fields (both amplitude and phase) on the surface $\phi = 0°$ at various distances (r) from the edge of a $\theta = 3\pi/2$ wedge; (b1, b2) thermal-wave fields on the surface $\phi = 0°$ of wedge-shaped structures with various wedge angles θ; and (c1, c2) frequency dependence of the thermal-wave field on the surface $\phi = 0°$ of a right edge (i.e., $\theta = \pi/2$) at various distances (r). *GF* Green's function method, *MI* method of images

It is seen that the thermal-wave field of a wedge under illumination by a Gaussian light beam is a complex function of thermophysical parameters of the material as well as geometrical factors of the solid. Detailed dependences of the thermal-wave field on different parameters must be calculated using a numerical method.

3 Numerical Calculations

To validate the theory, we consider R to be large enough so as to ignore the influence of the geometric discontinuity effects due to the presence of the edge (strictly true for $R \to \infty$). In addition, we use a uniform illumination beam (spot size W is large compared to the thermal diffusion length). AISI 304 stainless steel is assumed to be the material of the sample under investigation. The thermophysical parameters of AISI 304 stainless steel are $k = 16.3\,\mathrm{W \cdot m^{-1} \cdot K^{-1}}$, $\alpha = 4.1 \times 10^{-6}\,\mathrm{m^2 \cdot s^{-1}}$[10]. The amplitude and phase of the thermal-wave field on the $\phi = 0°$ surface of the wedge are normalized to those of a flat AISI 304 stainless steel.

Figure 2a1 and a2 shows the thermal-wave fields at various distances (r) from the edge on the $\phi = 0°$ surface. The wedge angle is $\theta = 3\pi/2$. From this figure, we can see that the effect of the corner on the thermal-wave field becomes stronger as the detection point moves closer to the corner, as expected. If the detection point is far away from the corner (i.e., r is large enough compared to the thermal diffusion length), the thermal-wave field reduces to that of a semi-infinite flat sample, as expected.

Different wedge angles will result in different thermal-wave distributions especially near the corner discontinuity along the z- axis ($r = 0$ region). Figure 2b1 and b2 shows the thermal-wave fields of structures with different wedge angles. The detection point is located at $\phi = 0°$ and $r = 1$ mm. It is seen that when $\theta = \pi/2$, i.e., the edge is a right-angle vertical wall, the normalized thermal-wave field (both amplitude and phase) coincides with that of a flat surface of a semi-infinite thickness. This result can be validated by comparing the results with those evaluated using the method of images [11], in which three imaging-point thermal sources are assumed to be symmetric to the inner point thermal source along two vertical walls and the origin of the corner to make homogeneous Neumann boundary conditions at the wall interfaces satisfied. The overall Green's function is a superposition of the four individual Green's function induced by those four point sources. Figure 2c1 and c2 shows the normalized amplitude and phase of the thermal-wave field of the same solid with a vertical wall ($\theta = \pi/2$) as that in Fig. 2b1 and b2 based on the method of images and the comparison with that obtained using the Green's function method. In Fig. 2c1 and c2, r is the distance from the corner to the detection point at the top surface, i.e., at surface $\phi = 0°$. From Fig. 2c1 and c2, it can be seen that both the amplitude and phase of these two theoretical models overlap completely. This is also the case with a semi-infinite flat surface in which the thermal-wave field has an invariable phase constant of $\pi/4$. This result is also the same as that obtained with the direct Green's function method for solids with a vertical edge [9].

4 Conclusions

In summary, we have developed an analytical expression for the thermal-wave field of arbitrary wedge-shaped structures using the Green's function method, following the derivation of the thermal-wave Green's function for adiabatic boundary conditions in this geometry. The thermal-wave field in wedge-shaped solids with irradiating Gaussian laser beams of an arbitrary spot size was obtained. The photothermal field model was validated theoretically by considering limiting cases of the wedge, such as right vertical walls and flat surface structures, and also comparing the results with well-known results from the method of images. This study offers a theoretical basis for photothermal characterization of wedge-shaped solids of industrial relevance.

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References

1. C. Wang, A. Mandelis, Y. Liu, J. Appl. Phys. **96**, 3756 (2004)
2. C. Wang, A. Mandelis, Rev. Sci. Instrum. **78**, 054902 (2007)
3. C. Wang, Y. Liu, A. Mandelis, J. Shen, J. Appl. Phys. **101**, 083503 (2007)
4. G. Xie, Z. Chen, C. Wang, A. Mandelis, Rev. Sci. Instrum. **80**, 0349031 (2009)
5. G. Xie, J. Zhang, L. Liu, C. Wang, A. Mandelis, J. Appl. Phys. **109**, 113534 (2011)
6. A. Salazar, F. Garrido, R. Celorrio, J. Appl. Phys. **99**, 066116 (2006)
7. A. Salazar, R. Celorrio, J. Appl. Phys. **100**, 113535 (2006)
8. N. Madariaga, A. Salazar, J. Appl. Phys. **101**, 103534 (2007)
9. A. Mandelis, Diffusion-Wave Fields: Mathematical Methods and Green Function (Springer, New York, 2001), pp. 90, 395, 483
10. J.N. Sweet, E.P. Roth, M. Moss, Int. J. Thermophys. **8**, 593 (1987)
11. L.C. Aamodt, J.C. Murphy, Appl. Opt. **21**, 111 (1982)