Theory of photothermal-wave diffraction and interference in condensed media

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Thermal-wave field diffraction has been treated as the extreme near-field approximation of a three-dimensional superposition integral that includes the generating optical aperture function. This formalism is quite general and is convenient for applications with many experimental diffracting apertures. Specific examples of useful photothermal excitation apertures have been treated explicitly. These include the spatial impulse function, a Gaussian laser beam, a circular aperture, and an expression for the interference field generated by two Gaussian laser beams.

1. INTRODUCTION

Photothermal-wave interferometry has received a certain degree of attention from the experimental community as a method for depth analysis and nondestructive evaluation of subsurface defects.[1–3] Experimental data obtained by several workers indicate that thermal-wave interference is possible and can be used to extract qualitative,[2] empirical,[3] or quasi-quantitative[1,3] material information. From the quantitative-analysis point of view, there remains, however, the need for putting photothermal-wave interferometry on a rigorous theoretical foundation that would explain interferometric phenomena as a result of the diffractive nature of photothermal waves. Besides the fact that such a theory is of great mathematical interest on its own merit, owing to the unusual, spatially damped nature of thermal waves, it should also be of practical interest to experimentalists: In the absence of rigorous theoretical guidance, they sometimes find it necessary to introduce arbitrary algebraic factors into their calculations in order to get the desirable fit to the data, as examplified in the paper by Sodnik and Tiziani.[2] It appears to the author that the lack of a proper theoretical basis for photothermal-wave diffraction analysis is related to the controversial nature of thermal waves as heavily damped pseudowaves (as discussed previously[4]) resulting from a specific form of the heat conduction equation and not from a proper wave equation. An additional difficulty stemming from this fact is that solutions to the thermal-wave field function (i.e., the temperature) must be considered in the extreme near-field approximation,[5] and thus the well-known Fresnel and/or Fraunhofer diffraction theories are not valid in this case.

In this paper the mathematical foundation for the photothermal-wave diffraction theory is developed. Under the experimentally justifiable condition of a small-aperture (SA) approximation, the diffraction integral is presented. Special cases are examined in a framework that is analogous to that of Fourier optics but is now valid for arbitrary aperture functions in convolution with the thermal-wave transfer function in what can be called, in analogy, Laplace thermal-wave physics.

2. PLANE THERMAL WAVES

On harmonic optical excitation of a material surface having the functional form

$$I(\mathbf{r}, t) = I_0(\mathbf{r})\exp(-i\omega t), \tag{1}$$

where I is the incident optical irradiance on the surface and $\omega = 2\pi f$ is the optical beam intensity modulation angular frequency, the resulting thermal field in the material can be described fully by the equation

$$\nabla^2 T(x, y, z) + \tilde{k}^2 T(x, y, z) = 0, \tag{2}$$

where T is the temperature field wave function and $\tilde{k}$ is the complex thermal-wave number given by

$$\tilde{k} = (1 + i)(\omega/2\alpha)^{1/2}. \tag{3}$$

In Eq. (3), α is the material thermal diffusivity. Equation (2) is valid under conditions of spatially invariant thermal conductivity. Owing to the complex nature of $\tilde{k}$, a more useful quantity in thermal-wave field theory is $k \equiv |\tilde{k}|$, where

$$k = (\omega/\alpha)^{1/2}. \tag{4}$$

Equation (4) helps us to define a thermal wavelength

$$\lambda_t = 2\pi(\alpha/\omega)^{1/2}. \tag{5}$$

It must be pointed out that this definition of λ_t, which is based on k, is slightly different from the conventional one,[4,6] which is based on

$$k_s(\omega) = k/\sqrt{2}, \tag{6}$$

where k_s is the thermal diffusion coefficient. Under these conditions, a unit-amplitude thermal plane wave may be constructed to describe the solution to the Helmholtz-like wave Eq. (2):

$$T(\mathbf{r}) = \exp(i\tilde{\mathbf{k}}_c \cdot \mathbf{r}), \tag{7}$$

where

$$\tilde{\mathbf{k}}_c \equiv \exp(i\pi/4)\tilde{k} = \left(\frac{1 + i}{\sqrt{2}}\right)(k_x\hat{i} + k_y\hat{j} + k_z\hat{k}). \tag{8}$$

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In Eq. (8) the set $(\hat{i}, \hat{j}, \hat{k})$ represents unit vectors in a Cartesian vector space. It is characteristic of the thermal-wave Helmoltz Eq. (2) that the wave-vector field $\tilde{\mathbf{k}}_c$ is rotated by 45° with respect to proper (real) wave vectors in the complex plane.

The field wave function Eq. (7) may be written explicitly as

$$T(x, y, z) = \exp[i \exp(i\pi/4)\hat{k}\cdot \mathbf{r}] = \exp[-\exp(-i\pi/4)k(\alpha_1 x + \alpha_2 y + \alpha_3 z)], \quad (9)$$

where $(\alpha_1, \alpha_2, \alpha_3)$ is the set of directional cosines, such that

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1. \quad (10)$$

Equation (9) can thus be written in the form

$$T(x, y, z) = T_0(x, y)\exp[-\exp(-i\pi/4)(2\pi/\lambda_t) \times (1 - \alpha_1^2 - \alpha_2^2)^{1/2} z], \quad (11)$$

where

$$T_0(x, y) \equiv \exp[-\exp(-i\pi/4)k(\alpha_1 x + \alpha_2 y)]. \quad (12)$$

It is useful to define thermal-wave spatial frequencies f_x, f_y, f_z by

$$f_x \equiv \alpha_1/\lambda_t, \qquad f_y \equiv \alpha_2/\lambda_t, \qquad f_z \equiv \alpha_3/\lambda_t. \quad (13)$$

Therefore, in terms of the spatial frequencies, a thermal-wave transfer function can also be defined as

$$H(f_x, f_y) \equiv \frac{T(x, y, z)}{T_0(x, y)} \quad (14)$$

and written as

$$H(f_x, f_y) = \exp\{-\exp(-i\pi/4)(2\pi/\lambda_t)[1 - \lambda_t^2(f_x^2 + f_y^2)]^{1/2} z\}, \quad (15)$$

analogous to the well-known optical transfer function.[7] Equation (15) indicates that the thermal-wave propagation requirement amounts to inclusion of spatial frequencies in the field spectrum up to λ_t^{-2}:

$$f_x^2 + f_y^2 < \lambda_t^{-2}. \quad (16)$$

3. THE THERMAL-WAVE DIFFRACTION INTEGRAL

At this point in the development of the theory, the approach deviates substantially from the standard Fourier integral diffraction formulation. The fact that thermal waves are of a heavily damped nature in the spatial parameter $\mathbf{r}$ forces us to consider an alternative spatial Laplace integral formalism, which physically represents Huygens's principle with exponentially damped thermal-wave propagation. If we define the two-dimensional complex spatial Laplace variables

$$s_x \equiv 2\pi \exp(-i\pi/4)f_x, \quad (17a)$$

$$s_y \equiv 2\pi \exp(-i\pi/4)f_y, \quad (17b)$$

the complex amplitude of the thermal-wave field across the xy plane at the surface ($z = 0$) can be written formally as[8]

$$T_0(x, y) = -\frac{1}{4\pi^2}\int_{c_1-i\infty}^{c_1+i\infty}\int_{c_2-i\infty}^{c_2+i\infty} A_0(s_x, s_y) \times \exp(s_x x + s_y y)\mathrm{d}s_x \mathrm{d}s_y, \quad (18)$$

where c_1, c_2 are appropriate real constants such that they can ensure convergence of the integrand at infinity. Furthermore, $T(x, y, z)$ can also be expressed in a similar form:

$$T(x, y, z) = -\frac{1}{4\pi^2}\int_{c_1-i\infty}^{c_1+i\infty}\int_{c_2-i\infty}^{c_2+i\infty} A(s_x, s_y, z) \times \exp(s_x x + s_y y)\mathrm{d}s_x \mathrm{d}s_y, \quad (19)$$

where $T(x, y, z)$ must satisfy the Helmholtz-like wave Eq. (2). Direct substitution of Eq. (19) into Eq. (2) yields the simple differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}A(s_x, s_y, z) + i(2\pi/\lambda_t)^2[1 - \lambda_t^2(f_x^2 + f_y^2)]A(s_x, s_y, z) = 0 \quad (20)$$

with a solution given in terms of Eq. (15),

$$A(s_x, s_y, z) = A_0(s_x, s_y)H(f_x, f_y), \quad (21)$$

where $A(s_x, s_y, 0) \equiv A_0(s_x, s_y)$. Equation (19) is a statement that $A(s_x, s_y, z)$ is the two-dimensional spatial Laplace transform of $T(x, y, z)$. The same relationship exists between $A_0(s_x, s_y)$ and $T_0(x, y)$ according to Eq. (18). Therefore Eq. (21) can be written as

$$A(s_x, s_y, z) = H(f_x, f_y)\int_0^\infty\int_0^\infty T_0(\zeta, \eta)\exp[-(s_x\zeta + s_y\eta)]\mathrm{d}\zeta \mathrm{d}\eta. \quad (22)$$

Substitution of Eq. (22) into the integrand of Eq. (19) results in the following convolution integral for $T(x, y, z)$:

$$T(x, y, z) = \int_0^\infty\int_0^\infty T_0(\zeta, \eta)G(x - \zeta, y - \eta)\mathrm{d}\zeta \mathrm{d}\eta, \quad (23)$$

where the kernel G is given by

$$G(x - \zeta, y - \eta) = -\frac{1}{4\pi^2}\int_{c_2-i\infty}^{c_2+i\infty} \exp[s_x(x - \zeta) + s_y(y - \eta)] \times \exp\{-\exp(-i\pi/4)kz[1 - \lambda_t^2(f_x^2 + f_y^2)]^{1/2}\}\mathrm{d}s_x \mathrm{d}s_y. \quad (24)$$

The integrand

$$H(f_x, f_y) \equiv \exp\{-\exp(-i\pi/4)kz[1 - \lambda_t^2(f_x^2 + f_y^2)]^{1/2}\}$$

is well behaved in the range $|f_x^2 + f_y^2| < 1/\lambda_t^2$:

$$|H(f_x, f_y)| = \exp\left\{-\frac{kz}{\sqrt{2}}[1 - \lambda_t^2(f_x^2 + f_y^2)]^{1/2}\right\}. \quad (25)$$

$H(f_x, f_y)$ is further bounded in the range $|f_x^2 + f_y^2| > 1/\lambda_t^2$:

$$|H(f_x, f_y)| = \exp\left\{-\frac{kz}{\sqrt{2}}[\lambda_t^2(f_x^2 + f_y^2) - 1]^{1/2}\right\} \to 0 \quad \text{as } f_x, f_y \to \infty. \quad (26)$$

Equation (25) and relation (26) thus show that the spatial-frequency strip of convergence for the transfer function $H(f_x, f_y)$ is $-\infty < (f_x^2 + f_y^2)^{1/2} < \infty$, and therefore $H(f_x, f_y)$ is analytic everywhere, with a branch cut extending between $-1 \le \lambda_t(f_x^2 + f_y^2)^{1/2} \le +1$. Therefore we can set $c_1 = c_2 = 0$ in

Eq. (24) and obtain the inverse transform of $H(f_x, f_y)$, namely, $G(x - \zeta, y - \eta)$, which will be convergent for all values $0 \le [(x - \zeta)^2 + (y - \eta)^2]^{1/2} < \infty$.[9] The kernel, Eq. (24), can be solved analytically after we make the variable transformations[10]

$$f_x = \rho \cos \phi, \qquad f_y = \rho \sin \phi, \tag{27a}$$

$$x - \zeta = r \cos \theta, \qquad y - \eta = r \sin \theta. \tag{27b}$$

In the new variables we obtain

$$G(r \cos \theta, r \sin \theta) = i \int_0^{\infty \exp(-i\pi/4)} \int_0^{\pi/2} \exp[-\exp(-i\pi/4) \times kz(1 - \lambda_t^2 \rho^2)^{1/2}] \exp[2\pi \exp(-i\pi/4)\rho r \cos(\phi - \theta)]\rho \mathrm{d}\rho \mathrm{d}\phi. \tag{28}$$

By using the representation (see Ref. 11, p. 958, entry 8.431.1)

$$I_0(x) = \frac{1}{\pi} \int_{-1}^{1} \frac{e^{xt}\mathrm{d}t}{(1 - t^2)^{1/2}} \tag{29}$$

for the modified Bessel function of the first kind, zeroth order, Eq. (28) can be written as

$$G(r) = \frac{i\pi}{2} \int_0^{\infty \exp(-i\pi/4)} \exp[-\exp(-i\pi/4)kz(1 - \lambda_t^2\rho^2)^{1/2}] \times I_0[2\pi \exp(-i\pi/4)\rho r]\rho \mathrm{d}\rho. \tag{30}$$

It is interesting to note that the angular dependence has dropped out of Eq. (30), leaving the kernel G as a circularly symmetric function of the radius r only:

$$G(r \cos \theta, r \sin \theta) \to G(r).$$

By using the well-known relations (see Ref. 11, p. 952, entry 8.406.1)

$$I_0(z) = J_0(iz), \qquad -\pi < \arg(z) \le \pi/2 \tag{31}$$

and letting $x = 2\pi \exp(i\pi/4)\rho$ in Eq. (30), we obtain

$$G(r) = \frac{1}{8\pi} \int_0^{\infty} \exp[-z(x^2 - ik^2)^{1/2}] J_0(rx) x \mathrm{d}x. \tag{32}$$

This integral can be evaluated explicitly by use of the relation (see Ref. 12, p. 95, entry 52)

$$Q(a, b) \equiv \int_0^{\infty} \exp[-a(t^2 - y^2)^{1/2}](t^2 - y^2)^{-1/2} J_0(bt) t \mathrm{d}t = \exp[-iy(a^2 + b^2)^{1/2}](a^2 + b^2)^{-1/2}, \quad \arg(t^2 - y^2) = \pi/2 \quad \text{if } t < y. \tag{33}$$

On taking $\partial Q(a, b)/\partial a$ we find that

$$\int_0^{\infty} \exp[-a(t^2 - y^2)^{1/2}] J_0(bt) t \mathrm{d}t = iya \frac{\exp[-iy(a^2 + b^2)^{1/2}]}{(a^2 + b^2)} \left[1 + \frac{1}{iy(a^2 + b^2)^{1/2}}\right]. \tag{34}$$

Let $a \equiv z$, $b \equiv r$, and $y \equiv -\exp(i\pi/4)k$; Eqs. (32) and (34) yield

$$G(r) = \frac{\exp(i\pi/4)}{4i\lambda_t} \left(\frac{z}{R_0}\right) \frac{\exp[-(1 - i)k_s(\omega)R_0]}{R_0} \times \left[1 + \frac{\exp(i\pi/4)}{kR_0}\right], \tag{35}$$

where $k_s(\omega) = (\omega/2\alpha)^{1/2}$, according to Eq. (6), and

$$R_0 \equiv (r^2 + z^2)^{1/2} = [(x - \zeta)^2 + (y - \eta)^2 + z^2]^{1/2}. \tag{36}$$

Equation (35) does converge for all values $0 < R_0 < \infty$, in agreement with the requirement for convergence of the inverse Laplace transform of $H(f_x, f_y)$, above. Now, in principle, Eq. (35) can be inserted into Eq. (23) to give an expression for the thermal-wave diffraction integral in terms of the photothermal aperture function $T_0(x, y)$. It is important to notice that the convolution integral

$$T(x, y, z) = T_0(x, y)^{**}G(x, y, z) \tag{23'}$$

is the two-dimensional Laplace transform of Eq. (21), with $G(r)$ being the spatial impulse response of the photothermal system. Furthermore, it can be shown rigorously in a manner entirely analogous to optical diffraction theory[7] that Eq. (23) with $G(x - \zeta, y - \eta)$ given by Eq. (35) is identical to the equation derived from Green's theorem by using Dirichlet boundary conditions on the aperture plane and

$$G_{k_c}^{(-)}(\mathbf{r}, \mathbf{r}_0) = \frac{\exp(i\tilde{k}R_0)}{R_0} - \frac{\exp(i\tilde{k}\tilde{R}_0)}{\tilde{R}_0}. \tag{37}$$

In Eq. (37), R_0 and $\tilde{R}_0$ have been defined as in Eq. (36), and they represent position vector magnitudes on either side of the diffracting aperture plane. $\tilde{k}$ was defined in Eq. (3).

This argument thus validates *a posteriori* the legitimacy of handling the pseudowave Eq. (2) as a proper Helmholtz wave equation with well-known Green's-function solutions. This mathematical equivalence is not generally valid for different classes of partial differential equations such as hyperbolic (the wave equation) and parabolic (the heat-diffusion equation) ones. In the spirit of this equivalence, the factor (z/R_0) in Eq. (35) can be interpreted as the thermal ray obliquity factor. In the case of thermal waves, however, the paraxial, Fresnel, and Fraunhofer approximations are generally not valid because of the heavily damped nature of the propagating field function $T(x, y, z)$. The field function must be evaluated in some extreme near-field approximation,[5] which is defined in Section 4. The wide-angle diffraction approximation[13] also exists and is a more restrictive form of the Fresnel approximation. Even this assumption, however, holds only for points of observation sufficiently far from the aperture plane. This condition is generally not valid for the thermal-wave field, and therefore the wide-angle diffraction approximation must also be abandoned.

4. THERMAL-WAVE DIFFRACTION: SMALL-APERTURE APPROXIMATION

In this section we discuss the extreme near-field approximation to the field integral

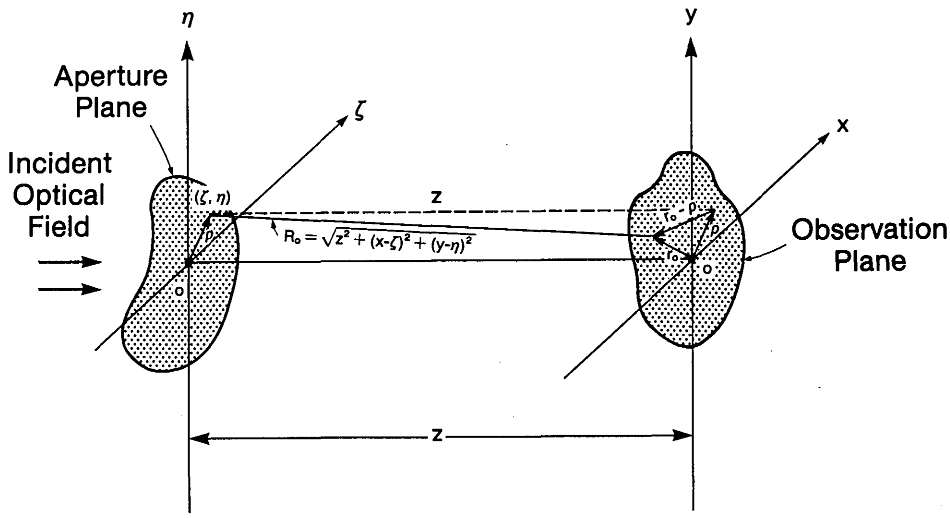

Fig. 1. Photothermal-wave diffraction geometry.

$$T(x, y, z) = \frac{\exp(i\pi/4)z}{4i\lambda_t} \int_0^\infty \int_0^\infty T_0(\zeta, \eta) \frac{\exp[-(1-i)k_s R_0]}{R_0^2} \times \left[1 + \frac{\exp(i\pi/4)}{kR_0}\right] d\zeta d\eta \quad (38)$$

in terms of the experimentally justifiable condition of a small photothermal aperture. We define variables r_1 and r_0 such that

$$\zeta^2 + \eta^2 = r_1^2 \quad \text{(aperture plane)} \quad (39)$$

and

$$x^2 + y^2 = r_0^2 \quad \text{(observation plane).} \quad (40)$$

The geometry of the field variables and position vectors is shown in Fig. 1. In terms of r_0, r_1, Eq. (36) can be written as

$$R_0 = [z^2 + (r_0 - r_1)^2]^{1/2}. \quad (41)$$

Experimentally, a large number of photothermal-wave investigations are done with focused (and even tightly focused) laser beams on material surfaces.[14] Under these conditions, the approximation will be valid for essentially all field positions outside the symmetry axis of the exciting laser beam. Expansion of the root in Eq. (41) gives, to first order,

$$R_0 \approx R - \frac{r_1 r_0}{R} + \frac{1}{2}\frac{r_1^2}{R}, \quad (43a)$$

where

$$R \equiv (x^2 + y^2 + z^2)^{1/2} = (r_0^2 + z^2)^{1/2}. \quad (43b)$$

The validity of this approximation depends on the magnitude of the next term in the expansion in relation to unity: the condition is met if the minimum probe distance R is such that

$$k_s(\omega) r_1^2 (2r_0 - r_1)^2 / 8R^3 \ll 1. \quad (44)$$

Assuming that $2r_0 \gg r_1$, condition (44) gives a worst-case requirement

$$\left(\frac{R^{3/2}}{r_0}\right)_{min} \gg \left[\frac{k_s(\omega)}{2}\right]^{1/2} (r_1)_{max}. \quad (45)$$

If $(r_1)_{max}$ is known, relation (45) can be viewed as the condition determining the minimum probe distance so that relation (43a) will be a valid representation of the thermal-wave field. If we use aluminum as a test material with $\alpha_s = 0.82$ cm^2/sec, $(r_1)_{max} = 50$ μm, $z = 1.5$ mm, and $(r_0)_{min} = 200$ μm and a laser-beam modulation frequency $f = 100$ Hz, we find that $(R^{3/2}/r_0)_{min} = 0.625$ cm$^{1/2}$ and $[k_s(100 \text{ Hz})/2]^{1/2} (r_1)_{max} = 0.016$ cm$^{1/2}$. These worst-case calculations show that the SA approximation (43a) is valid for a broad range of materials and experimental conditions.

For the rest of this paper the assumption of thermally thick solids is adopted[15]:

$$k_s(\omega) R_0 \approx \frac{1}{\sqrt{2}} kR \gg 1, \quad (46)$$

so that in Eq. (38)

$$1 + \frac{\exp(i\pi/4)}{kR_0} \approx 1. \quad (47)$$

The circular symmetry built into the diffraction integral Eq. (38) points to polar coordinates as the most convenient representation. In the limit of the SA approximation,

$$R_0 \approx R - \frac{1}{R}(x\zeta + y\eta) + \frac{1}{2R}(\zeta^2 + \eta^2), \quad (48)$$

so that

$$T(x, y, z) = \frac{\exp(i\pi/4)}{4i\lambda_t}\left(\frac{z}{R}\right)\frac{\exp[-(1-i)k_s R]}{R} \times \int_0^\infty \int_0^\infty (T_0(\zeta, \eta)\exp\{-[(1-i)k_s/2R](\zeta^2 + \eta^2)\}) \times \exp\{[(1-i)k_s/R](x\zeta + y\eta)\} d\zeta d\eta. \quad (49)$$

In accordance with Eqs. (39) and (40) we define the observation-plane variables

$$x = r_0 \cos \Psi_0, \quad (50)$$

$$y = r_0 \sin \Psi_0, \quad (51)$$

and the aperture-plane variables

$$\zeta = r_1 \cos \Psi_1, \quad (52)$$

$$\eta = r_1 \sin \Psi_1. \quad (53)$$

Further, we define the complex polar-plane Laplace variable

$$s \equiv 2\pi \exp(-i\pi/4) f_{r_0}, \qquad f_{r_0} \equiv \frac{r_0}{\lambda_t R}, \quad (54)$$

where f_{r_0} is a polar thermal-wave spatial frequency. Finally, for purposes of conforming with convention regarding the representation of Laplace transforms, we let $(x, y) \rightarrow (-x, -y)$ in Eq. (49). This operation leaves the value of the circularly symmetric integral unchanged. It simply moves the two-dimensional complex spatial Laplace plane domain to the third quadrant instead of the first quadrant. Using these definitions, we can write, instead of Eq. (49),

$$T(x, y, z) \rightarrow T(r_0, \Psi_0, z) = \frac{\exp(i\pi/4)}{4i\lambda_t}\left(\frac{z}{R}\right) \times \frac{\exp[-(1-i)k_s R]}{R} K(r_0, \Psi_0, z), \quad (55)$$

where R is given by Eq. (43b) and

$$K(r_0, \Psi_0, z) = \int_0^\infty \int_0^{\pi/2} (T_0(r_1, \Psi_1)\exp\{-[(1-i)k_s/2R]r_1^2\}) \times \exp[-s r_0 r_1 \cos(\Psi_1 - \Psi_0)] r_1 dr_1 d\Psi_1. \quad (55')$$

Assuming circularly symmetric aperture functions,

$$T_0(r_1, \Psi_1) = T_0(r_1), \quad (56)$$

and using Eq. (29) and relations (31), we find that

$$K(r_0, \Psi_0, z) = K(r_0, z) = \frac{\pi}{2}\int_0^\infty \{T_0(r_1)\exp[-(s/2r_0)r_1^2]\} J_0(isr_1) r_1 dr_1. \quad (57)$$

Equation (57) is the Laplace–Bessel transform of the function within the braces (see Appendix A):

$$K(r_0, z) = {}^2L_B\{T_0(r_1)\exp[-(s/2r_0)r_1^2]\}|_{\rho=s}. \quad (58)$$

5. EXAMPLES OF SMALL-APERTURE DIFFRACTION PATTERNS

We consider next a few special cases of the thermal-wave diffraction integral [relation (55)] that correspond to useful experimental configurations with optical excitation of the material surface (aperture plane, Fig. 1). These cases are classified in terms of the functional form of the optically generated photothermal source function $T_0(r_1)$. For many spatial functions, solutions to the problem of relation (55) and Eq. (58) are readily available in tables of two-dimensional Hankel transforms to which the Laplace–Bessel integral bears close similarity.

A. The Spatial Impulse: $T_0(r_1) = \delta(r_1)/\pi r_1$

This is the simplest form of optical excitation and is relevant in situations when a pump laser beam is focused tightly upon the sample surface. Equation (58) readily gives

$$K(r_0, z) = {}^2L_B\left\{\frac{\delta(r_1)}{\pi r_1}\exp[-s/2r_0)r_1^2\right\}_{\rho=s} = 1. \quad (59)$$

The field function T [relation (55)] thus becomes

$$T(r_0, z) = \frac{\exp(i\pi/4)}{4i\lambda_t}\left(\frac{z}{R}\right)\frac{\exp[-(1-i)k_s R]}{R}. \quad (60)$$

Equation (60) is, in fact, the spatial impulse response itself and is identical to the function derivable from the exact expression, Eq. (38) in the thermally thick limit, under impulse excitation. Therefore, in the limit of infinitesimally small photothermal sources, the SA approximation gives the exact solution, as expected.

B. A Gaussian Laser-Beam Profile: TEM_{00} Mode

Let

$$T_0(r_1) = \exp(-r_1^2/w^2). \quad (61)$$

Equation (61) describes a laser photothermal excitation of unit amplitude and a beam waist of size w; then

$$K(r_0, z) \equiv Q(\rho)|_{\rho=s}, \quad (62)$$

where

$$Q(\rho) = {}^2L_B\{\exp[-(s/2r_0 + w^{-2})r_1^2]\} = \frac{\pi}{2}\int_0^\infty \exp(-Br_1^2) J_0(i\rho r_1) r_1 dr_1, \quad (63)$$

with

$$B \equiv \frac{s}{2r_0} + \frac{1}{w^2}. \quad (64)$$

Now, we use the result (see Ref. 11, p. 716, entry 6.631)

$$\int_0^\infty \exp(-\alpha x^2) J_0(\beta x) x dx = \frac{\exp(-\beta^2/8\alpha)}{\beta\sqrt{\alpha}} M_{1/2,0}\left(\frac{\beta^2}{4\alpha}\right), \quad (65)$$

where $M_{1/2,0}$ is the Whittaker function defined by

$$M_{1/2,0}(q) = z^{1/2}\exp(-z/2)|_{z=q} = q^{1/2}\exp(-q/2). \quad (66)$$

Therefore we obtain

$$Q(\rho) = \frac{\pi}{4B}\exp(-\rho^2/4B), \quad (67)$$

so that

$$K(r_0, z) = \frac{\pi}{4B}\exp(-s^2/4B). \quad (68)$$

Substituting Eqs. (54) and (64) into Eq. (68) and separating out real and imaginary parts, we obtain

$$K(r_0, z) = \frac{\pi}{4(F_1^2 + F_2^2)^{1/2}}\exp\left[-\frac{(k_s r_0/R)^2 F_2}{2(F_1^2 + F_2^2)}\right] \times \exp\left\{i\left[\frac{(k_s r_0/R)^2 F_1}{2(F_1^2 + F_2^2)} + \tan^{-1}(F_2/F_1)\right]\right\}, \quad (69)$$

where

$$F_1(R, w) \equiv \frac{1}{w^2} + \frac{k_s(\omega)}{2R} \quad (70)$$

and

$$F_2(R) \equiv \frac{k_s(\omega)}{2R}. \quad (71)$$

Finally, when Eq. (69) is substituted into relation (55), we obtain the full expression for the thermal-wave field, which can be written out in terms of its experimentally convenient components, amplitude and phase, as follows:

$$|T(r_0, z)| = \frac{1}{16\sqrt{2}}\left[\frac{k_s z}{R^2(F_1{}^2 + F_2{}^2)^{1/2}}\right] \times \exp\left\{-\left[k_s R + \frac{(k_s/R)^2 F_2}{2(F_1{}^2 + F_2{}^2)} r_0{}^2\right]\right\} \quad (72)$$

and

$$\Psi(r_0, z) = -\frac{\pi}{4} + k_s R + \frac{(k_s r_0/R)^2 F_1}{2(F_1{}^2 + F_2{}^2)} + \tan^{-1}(F_2/F_1). \quad (73)$$

C. A Circular Aperture of Uniform Irradiance: $T_0(r_1) = \mathrm{circ}(r_1/L)$

A circular aperture of uniform irradiance occurs in the case of material irradiation with an optical field of uniform intensity, such as a laser beam, passed through a beam expander followed by a pinhole, or the light generated by a spectral lamp producing a uniform spatial intensity. We assume a surface temperature field of unit magnitude and aperture radius L; then

$$Q(\rho) = {}^2L_B\{\exp[-(s/2r_0)r_1{}^2]\mathrm{circ}(r_1/L)\}. \quad (74)$$

Equation (74) can be written in the form

$$Q(\rho) = \frac{\pi}{2}\int_0^L \exp[-(s/2r_0)r_1{}^2]J_0(i\rho r_1)r_1 \mathrm{d}r_1. \quad (75)$$

Using Lommel's functions U_1 and U_2, defined as[16]

$$U_1(a, w) = a\int_0^1 \cos\left[\frac{a}{2}(1 - r^2)\right]J_0(wr)r\mathrm{d}r, \quad (76a)$$

$$U_2(a,w) = a\int_0^1 \sin\left[\frac{a}{2}(1 - r^2)\right]J_0(wr)r\mathrm{d}r, \quad (76b)$$

we obtain the composite formula

$$\int_0^1 \exp(-iar^2/2)J_0(wr)r\mathrm{d}r = \frac{1}{a}(U_1 + iU_2)\exp(-ia/2). \quad (77)$$

From Eqs. (75) and (77), after substituting $x \equiv r_1/L$ and $a \equiv -isL^2/r_0 = -\exp(i\pi/4)kL^2/R$, and $w \equiv i\rho L$, we obtain

$$Q(\rho) = -\frac{\exp(-i\pi/4)}{4}(\lambda_t R)\exp[-(1 - i)k_s L^2/2R] \times \{U_1[-\exp(i\pi/4)kL^2/R, i\rho L] + iU_2[-\exp(i\pi/4)kL^2/R, i\rho L]\}. \quad (78)$$

Use of Eq. (62), relation (55), and the properties of Lommel's functions,

$$U_1(-a, w) = -U_1(a, w), \quad (79a)$$

$$U_2(-a, w) = U_2(a, w), \quad (79b)$$

yields an expression for the photothermal-wave field under uniform illumination of a circular aperture:

$$T(r_0, z) = \frac{-i}{16}\left(\frac{z}{R}\right)\exp\left[-(1 - i)k_s\left(R + \frac{L^2}{2R}\right)\right] \times \{U_1[\exp(i\pi/4)kL^2/R, \exp(i\pi/4)kLr_0/R] - iU_2[\exp(i\pi/4)kL^2/R, \exp(i\pi/4)kLr_0/R]\}. \quad (80)$$

For computational purposes, Lommel's functions may be expressed in terms of infinite series (see Ref. 17, p. 537, entry 1):

$$U_n(a, w) = \sum_{m=0}^{\infty}(-1)^m\left(\frac{a}{w}\right)^{n+2m} J_{n+2m}(w). \quad (81)$$

In particular, we have

$$U_1[\exp(i\pi/4)kL^2/R, \exp(i\pi/4)kLr_0/R] = \sum_{m=0}^{\infty}(-1)^m(L/r_0)^{2m+1}J_{2m+1}[\exp(i\pi/4)kLr_0/R] \quad (82)$$

and

$$U_2[\exp(i\pi/4)kL^2/R, \exp(i\pi/4)kLr_0/R] = \sum_{m=0}^{\infty}(-1)^m(L/r_0)^{2m+2}J_{2m+2}[\exp(i\pi/4)kLr_0/R], \quad (83)$$

with the integral-order Bessel function defined by

$$J_{n+2m}[\exp(i\pi/4)kLr_0/R] = \left(\frac{kLr_0}{2R}\right)^{n+2m}\exp[i(n + 2m)\pi/4] \times \sum_{p=0}^{\infty}\frac{(-1)^p}{p!(n + 2m + p)!}\left(\frac{kLr_0}{2R}\right)^{2p}\exp(ip\pi/2). \quad (84)$$

Now, Eqs. (80)–(84) can be separated out into real and imaginary components to yield the experimentally observable amplitude and phase of the photothermal field.

The present diffraction theory makes possible the quantitative evaluation of the photothermal-wave field at specific infinitesimally small probe points within a material of interest under arbitrary excitation apertures. Its experimental implementation, however, relies on probes of finite size, which will have an integrating effect of the field values over the probe area. These effects are currently under investigation in the light of criteria imposed by the Whittaker–Shannon sampling theorem.[18,19] Appendix B shows the effects, in the simplest possible limit, of an integrating detector of infinite dimensions on the $T(x, y, z)$ field generated by a spatial photothermal impulse in terms of a complete loss of resolution in radial directions.

6. PHOTOTHERMAL-WAVE INTERFEROMETRY

In this section we present the theory of photothermal-wave interferometry based on the diffraction formalism developed in Sections 3 and 4 in the SA approximation. As an

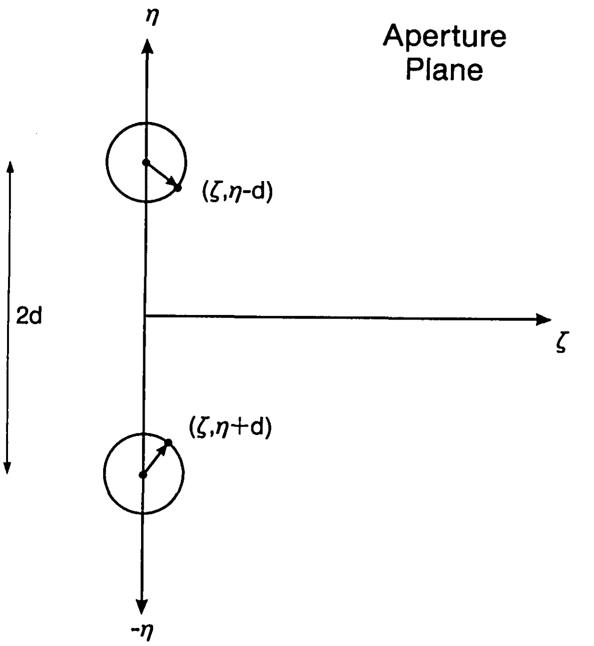

Fig. 2. Photothermal-wave interference geometry on the aperture plane (ζ, η).

experimental geometry we use the photothermal effects produced by two Gaussian laser beams impinging upon a material surface in or out of phase with respect to intensity modulation. This geometry is shown in Fig. 2 and has the additional virtue that it essentially represents published experimental configurations as well.[1,2] The theoretical description of two diffracting apertures producing interfering thermal-wave patterns requires a shift of the origin of the polar coordinate system. This operation unfortunately destroys the circular symmetry on which the Laplace–Bessel transform formulation is based. Therefore it is necessary to revert to Cartesian coordinates [approximation (48) and Eq. (49)]. The Laplace variables s_x and s_y are given by Eqs. (17a) and (17b) with spatial frequencies defined by

$$f_x \equiv \frac{x}{\lambda_t R}, \qquad f_y \equiv \frac{y}{\lambda_t R}. \tag{85}$$

With these definitions, Eq. (49) can be written as

$$T(x, y, z) = \frac{\exp(i\pi/4)}{4i\lambda_t}\left(\frac{z}{R}\right)\frac{\exp[-(1-i)k_s R]}{R} \times \int_0^\infty \int_0^\infty [T_0(\zeta, \eta)\exp(-\mathbf{P}\cdot\boldsymbol{\xi})]\exp[-(s_x\zeta + s_y\eta)]d\zeta d\eta, \tag{86}$$

where the double integral was translated to the third quadrant in the complex Laplace plane for proper convergence: $(x, y) \rightarrow (-x, -y)$. In Eq. (86) we defined $\mathbf{P} = P_x\hat{\zeta} + P_y\hat{\eta}$, where

$$P_x \equiv (1-i)k_s/2R = s_x/2x, \qquad P_y \equiv (1-i)k_s/2R = s_y/2y, \tag{87a}$$

and

$$\boldsymbol{\xi} = \zeta^2\hat{\zeta} + \eta^2\hat{\eta}, \tag{87b}$$

so that the thermal-wave field can be expressed as a two-dimensional complex Laplace transform (see Appendix A):

$$T(x, y, z) = \frac{\exp(i\pi/4)}{4i\lambda_t}\left(\frac{z}{R}\right)\frac{\exp[-(1-i)k_s R]}{R} \times {}^2L[T_0(\zeta, \eta)\exp(-\mathbf{P}\cdot\boldsymbol{\xi})]_{\substack{s_x = 2\pi\exp(-i\pi/4)f_x \\ s_y = 2\pi\exp(-i\pi/4)f_y}}. \tag{88}$$

Since the geometry of Fig. 2 is a generalization of the single-aperture geometry, we consider first the Cartesian representation of the photothermal-wave field of a TEM_{00} Gaussian laser-beam profile of unit amplitude:

$$T_0(\zeta, \eta) = \exp[-(\zeta^2 + \eta^2)/w^2]. \tag{89}$$

Now, let

$$Q(s_x, s_y) \equiv {}^2L[T_0(\zeta, \eta)\exp(-\mathbf{P}\cdot\boldsymbol{\xi})] = {}^2L\{\exp[-(P_x\zeta^2 + P_y\eta^2) - (\zeta^2 + \eta^2)/w^2]\} \tag{90}$$

$$= L\{\exp[-(P_x + w^{-2})\zeta^2]\}L\{\exp[-(P_y + w^{-2})\eta^2]\}. \tag{91}$$

The relation (see Ref. 22, p. 146, entry 24)

$$L[\exp(-u^2/8\alpha)] = 2\alpha^{1/2}\exp(\alpha s^2)D_{-1}(2\alpha^{1/2}s) \tag{92}$$

can be used, where D_{-1} is a parabolic cylinder function of order -1, given by (see Ref. 11, p. 1067, entry 9.254.1)

$$D_{-1}(z) = \left(\frac{\pi}{2}\right)^{1/2}\exp(z^2/4)\mathrm{erfc}(z/\sqrt{2}). \tag{93}$$

Equations (92) and (93), when adapted to Eq. (91), yield

$$Q(s_x, s_y) = \frac{\pi}{4[(P_x + w^{-2})(P_y + w^{-2})]^{1/2}} Z\left[\frac{{s_x}^2}{4(P_x + w^{-2})}\right] \times Z\left[\frac{{s_y}^2}{4(P_y + w^{-2})}\right], \tag{94}$$

where we have defined

$$Z(x) \equiv e^x \,\mathrm{erfc}\sqrt{x}. \tag{95}$$

Finally, Eqs. (88) and (94) determine the field:

$$T(x, y, z) = \frac{\exp(-i\pi/4)k_s z}{4\sqrt{2}\pi R^2} Q(s_x, s_y)\Bigg|_{\substack{s_x = 2\pi\exp(-i\pi/4)f_x \\ s_y = 2\pi\exp(-i\pi/4)f_y}}. \tag{96}$$

Now, turning our attention to the photothermal field generated by the geometry of Fig. 2, we assume that two laser beams are incident upon the surface, with a center-to-center distance $2d$ along the η axis of the aperture plane. Both beams are of equal intensity and thus generate photothermal wave fields of equal (unit) amplitudes in the sample.

Assuming equal spatial spot sizes ($w_1 = w_2 = w$) and in-phase operation, we obtain the photothermal-field equivalent of the well-known optical-field Young experiment. Under these conditions

$$T_0^{(+)}(\zeta, \eta) = \exp\{-[\zeta^2 + (\eta - d)^2]/w^2\} + \exp\{-[\zeta^2 + (\eta + d)^2]/w^2\}. \tag{97}$$

Therefore

$$Q_2(s_x, s_y) \equiv L\{\exp[-(P_x + w^{-2})\zeta^2]\}(L\{\exp[-P_y\eta^2 - (\eta - d)^2w^2]\} + L\{\exp[-P_y\eta^2 - (\eta + d)^2/w^2]\}). \tag{98}$$

Expanding the exponents of the η-dependent transforms gives the relations

$$L\{\exp[-P_y\eta^2 - (\eta - d)^2/w^2]\} = \exp(-d^2/w^2)L\{\exp[-(P_y + w^{-2})\eta^2]\}_{s_y - 2d/w^2},$$

$$L\{\exp[-P_y\eta^2 - (\eta + d)^2/w^2]\} = \exp(-d^2w^2)L\{\exp[-(P_y + w^{-2})\eta^2]\}_{s_y + 2d/w^2}. \quad (99)$$

By combining Eqs. (98) and (99) and using Eq. (91), we obtain a simple relation between $Q(s_x, s_y)$ and $Q_2(s_x, s_y)$:

$$Q_2(s_x, s_y) = \exp(-d^2/w^2)[Q(s_x, s_y - 2d^2/w^2) + Q(s_x, s_y + 2d^2/w^2)]. \quad (100)$$

$Q(s_x, s_y)$ is given explicitly by Eq. (94), while Eq. (100) represents a form of the shift property of the two-dimensional Laplace transform.[8] Finally, we can write the complex thermal-wave field expression for the two constructively interfering photothermal sources as follows:

$$T^{(+)}(x, y, z) = \frac{\exp(-i\pi/4)k_s z}{16\sqrt{2}R^2} \frac{\exp\{-[(1 - i)k_s R + d^2/w^2]\}}{F_1 - iF_2} \times Z\left[-i\frac{(k_s/R)^2x^2}{2(F_1 - iF_2)}\right] \times \left(Z\left\{\frac{\left[\left(F_2 y - \frac{d}{w^2}\right) - iF_2 y\right]^2}{F_1 - iF_2}\right\} + Z\left\{\frac{\left[\left(F_2 y + \frac{d}{w^2}\right) - iF_2 y\right]^2}{F_1 - iF_2}\right\}\right), \quad (101)$$

where $F_1(R, w)$ and $F_2(R)$ are given by Eqs. (70) and (71), respectively.

The conditions for thermal-wave destructive interference can be generated with out-of-phase operation of the laser beams, in which case

$$T_0^{(-)}(\zeta, \eta) = \exp\{-[\zeta^2 + (\eta - d)^2]/w^2\} - \exp\{-[\zeta^2 + (\eta + d)^2]/w^2\}. \quad (102)$$

This aperture function results in the interference pattern

$$T^{(-)}(x, y, z) = \frac{\exp(-i\pi/4)k_s z}{16\sqrt{2}R^2} \frac{\exp\{-[(1 - i)k_s R + d^2/w^2]\}}{F_1 - iF_2} \times Z\left[-i\frac{(k_s/R)^2x^2}{2(F_1 - iF_2)}\right] \times \left(Z\left\{\frac{\left[\left(F_2 y - \frac{d}{w^2}\right) - iF_2 y\right]^2}{F_1 - iF_2}\right\} - Z\left\{\frac{\left[\left(F_2 y + \frac{d}{w^2}\right) - iF_{2y}\right]^2}{F_1 - iF_2}\right\}\right). \quad (103)$$

Note that $[T^{(+)}(x, y, z; d)]_{\max} = T^{(+)}(x, y, z; 0)$ and $[T^{(-)}(x, y, z; d)]_{\min} = T^{(-)}(x, y, z; 0) = 0$, as expected.

Separating out real and imaginary parts in Eqs. (101) and (103) gives a convenient representation leading to amplitude and phase field components as follows:

$$T^{(\pm)}(x, y, z) = \frac{k_s z \exp[-(k_s R + d^2/w^2)]}{16\sqrt{2}R^2(F_1{}^2 + F_2{}^2)^{1/2}} \times \exp\left\{i\left[k_s R - \frac{\pi}{4} + \tan^{-1}(F_2/F_1)\right]\right\} \times \exp(z_1{}^2)\operatorname{erfc}(z_1)[\exp(z_2{}^2)\operatorname{erfc}(z_2) \pm \exp(z_3{}^2)\operatorname{erfc}(z_3)], \quad (104)$$

where

$$z_1 \equiv |z_1|\exp(i\theta_1),$$

with

$$|z_1| = \frac{k_s x}{\sqrt{2}R(F_1{}^2 + F_2{}^2)^{1/4}}, \qquad \theta_1 = \frac{1}{2}\tan^{-1}(F_2/F_1) - \frac{\pi}{4}; \quad (105)$$

also

$$z_2 \equiv |z_2|\exp(i\theta_2),$$

with

$$|z_2| = \frac{(F_3{}^2 + F_4{}^2)^{1/2}}{2(F_1{}^2 + F_2{}^2)^{1/4}},$$

$$\theta_2 = {}^1/_2 \tan^{-1}(F_2/F_1) - \tan^{-1}(F_4/F_3), \quad (106)$$

where we defined

$$F_3(x, y, z; d) \equiv \frac{k_s y}{R} - \frac{2d}{w^2}, \quad (107)$$

$$F_4(x, y, z) \equiv \frac{k_s y}{R}. \quad (108)$$

Finally, we obtain

$$z_3 \equiv |z_3| \exp(i\theta_3),$$

with

$$|z_3| = \frac{(F_5{}^2 + F_4{}^2)^{1/2}}{2(F_1{}^2 + F_2{}^2)^{1/4}},$$

$$\theta_3 = {}^1/_2 \tan^{-1}(F_2/F_1) - \tan^{-1}(F_4/F_5), \quad (109)$$

where

$$F_5(x, y, z; d) \equiv \frac{k_s y}{R} + \frac{2d}{w^2}. \quad (110)$$

If we set

$$W(x) \equiv Z(x^2) = \exp(x^2)\operatorname{erfc}(x), \quad (111)$$

then the amplitude of the field is found to be

$$|T^{(\pm)}(x, y, z)| = \frac{k_s z|N_1|\,|N_2|^{(\pm)}}{16\sqrt{2}R^2(F_1{}^2 + F_2{}^2)^{1/2}} \exp\left[-\left(k_s R + \frac{d^2}{w^2}\right)\right], \quad (112)$$

where

$$|N_1| \equiv \{[\mathrm{Re}\ W(z_1)]^2 + [\mathrm{Im}\ W(z_1)]^2\}^{1/2} \quad (113a)$$

and

$$|N_2|^{(\pm)} \equiv \{[\mathrm{Re}\ W(z_2) \pm \mathrm{Re}\ W(z_3)]^2 + [\mathrm{Im}\ W(z_2) \pm \mathrm{Im}\ W(z_3)]^2\}^{1/2}. \quad (113b)$$

The phase of the field can then be written as

$$\Psi^{(\pm)}(x, y, z) = -\frac{\pi}{4} + k_s R + \tan^{-1}(F_2/F_1) + \phi_1 + \phi_2^{(\pm)}, \quad (114)$$

where

$$\phi_1 \equiv \tan^{-1}\left[\frac{\mathrm{Im}\ W(z_1)}{\mathrm{Re}\ W(z_1)}\right] \quad (115a)$$

and

$$\phi_2^{(\pm)} \equiv \tan^{-1}\left[\frac{\mathrm{Im}\ W(z_2) \pm \mathrm{Im}\ W(z_3)}{\mathrm{Re}\ W(z_2) \pm \mathrm{Re}\ W(z_3)}\right]. \quad (115b)$$

For computational purposes, series expressions for Re $W(z_j)$ and Im $W(z_j)$ can be found readily in the appendix of Ref. 20 and thus are not repeated here.

7. CONCLUSIONS

In this paper we have developed a photothermal-wave diffraction formalism describing the temperature field dependence on arbitrary diffracting aperture geometries. Special cases of experimental importance were then examined in detail, including source geometries leading to constructive or destructive thermal-wave interference. It is expected that the present theory will help to quantify experimental observations encountered in photothermal-wave imaging and interferometry.

APPENDIX A: POLAR-COORDINATE REPRESENTATION OF TWO-DIMENSIONAL SPATIAL LAPLACE TRANSFORM (LAPLACE-BESSEL TRANSFORM)

Let $G(s_x, s_y) = \int_0^\infty \int_0^\infty g(x, y)\exp[-(s_x x + s_y y)]\mathrm{d}x\mathrm{d}y$ be the two-dimensional spatial Laplace transform of $g(x, y)$.[8] Using polar coordinates,

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

$$s_x = \rho\cos\phi, \qquad s_y = \rho\sin\phi, \quad (A1)$$

and assuming circular symmetry for the function g,

$$g(x, y) \to g(r, \theta) = g_R(r), \quad (A2)$$

yields

$$G(\rho, \phi) = G(\rho) = \int_0^\infty g_R(r)r\mathrm{d}r \int_0^1 \frac{\exp(-r\rho t)}{(1 - t^2)^{1/2}}\mathrm{d}t = \frac{\pi}{2}\int_0^\infty g_R(r)I_0(\rho r)r\mathrm{d}r. \quad (A3)$$

Finally, use of relations (31) (see also Ref. 11, p. 952, entry 8.406.2) gives the circularly symmetric Laplace–Bessel transform,

$$G(\rho) = \frac{\pi}{2}\int_0^\infty g_R(r)J_0(\pm i\rho r)r\mathrm{d}r, \quad (A4)$$

with $(+i)$ used if $-\pi < \arg(\rho) \leq \pi/2$ and with $(-i)$ used if $\pi/2 < \arg(\rho) \leq \pi$. Symbolically one may write

$$^2L_B[g_R(r)] = G(\rho). \quad (A5)$$

Note that $G(\rho)$ is defined as one fourth of the polar Fourier–Bessel (or Hankel) transform.[21] This is so because only one quarter of the plane is pertinent to the two-dimensional Laplace transformation: $0 \leq \phi \leq \pi/2$.

APPENDIX B: PHOTOTHERMAL-FIELD INTEGRATION EFFECTS CAUSED BY A DETECTOR OF INFINITE APERTURE

For simplicity we will assume that the diffraction field is generated by a photothermal source of the spatially impulsive type; then, according to Eq. (60),

$$T(r_0, z) = \left[\frac{\exp(i\pi/4)z}{4i\lambda_t}\right]\frac{\exp[-(1 - i)k_s R]}{R^2}. \quad (B1)$$

A detector of active area A placed in contact with the sample at a depth $z = z_0$, which may indicate the thickness of the sample, will produce an integrated signal of the form

$$S_{\mathrm{detector}} = \int_A\int T(r_0, z)\mathrm{d}A. \quad (B2)$$

If we let $A \to \infty$, then Eqs. (B1) and (B2) yield

$$S_{\mathrm{detector}}(z_0) = \frac{\exp(i\pi/4)k_s z_0}{4i} I(z_0), \quad (B3)$$

where

$$I(z_0) \equiv \int_0^\infty \frac{\exp[-(1 - i)k_s(r_0^2 + z_0^2)^{1/2}]}{(r_0^2 + z_0^2)} r_0\mathrm{d}r_0 = \frac{1}{z_0^2}\int_0^\infty \frac{\exp\{-(1 - i)k_s z_0[1 + (r_0/z_0)^2]^{1/2}\}}{1 + (r_0/z_0)^2} r_0\mathrm{d}r_0. \quad (B4)$$

This integral can be evaluated explicitly by using the relation (see Ref. 12, p. 83, entry 30)

$$\int_0^\infty \frac{x^{-2\nu}\exp[iz(1 + x^2)^{1/2}]}{(1 + x^2)}\mathrm{d}x = \frac{i\sqrt{\pi}}{2}\left(\frac{z}{2}\right)^\nu \Gamma\left(\frac{1}{2} - \nu\right)H_\nu^{(1)}(z),$$

$$\mathrm{Im}(z) > 0, \qquad \mathrm{Re}(\nu) < 1/2, \quad (B5)$$

with

$$H_\nu^{(1)}(z) = J_\nu(z) + iY_\nu(z) = \frac{1}{i\sin(\nu\pi)}[J_{-\nu}(z) - J_\nu(z)\exp(-i\nu\pi)]. \quad (B6)$$

J_ν and Y_ν are Bessel and Neumann functions, respectively, of order ν. If we let $\nu = -1/2$, then relations (B5) give

$$Q(z) \equiv \int_0^\infty \frac{\exp[iz(1+x^2)^{1/2}]}{(1+x^2)^{1/2}}\, x\mathrm{d}x = i\,\frac{\sqrt{\pi}}{2}\left(\frac{2}{z}\right)^{1/2} H_{-1/2}^{(1)}(z), \tag{B7}$$

where, from Eq. (B6) and properties of the Bessel function, it can be shown that

$$H_{-1/2}^{(1)}(z) = \left(\frac{2}{\pi z}\right)^{1/2} e^{iz}. \tag{B8}$$

Thus we conclude that

$$Q(z) = \frac{ie^{iz}}{z}. \tag{B9}$$

Now, by integrating Eq. (B9) and taking Eq. (B7) into account, we obtain

$$\int_z^\infty Q(y)\mathrm{d}y = \int_0^\infty \frac{x\mathrm{d}x}{(1+x^2)^{1/2}} \int_z^\infty \exp[iy(1+x^2)^{1/2}]\mathrm{d}y$$

$$= i\int_0^\infty \frac{x\mathrm{d}x}{1+x^2}\{\exp[iy(1+y^2)^{1/2}]_z^\infty\}. \tag{B10}$$

We must note that y is a complex argument of the form $y_1 + iy_2$ with $y_2 > 0$. Under this condition, we have

$$\lim_{y_2\to\infty} \exp[iy(1+x^2)^{1/2}] = 0, \tag{B11}$$

and Eq. (B10) gives

$$\int_0^\infty \frac{\exp[iz(1+x^2)^{1/2}]}{1+x^2}\, x\mathrm{d}x = -\int_z^\infty \frac{e^{iy}}{y}\mathrm{d}y. \tag{B12}$$

On setting $x \equiv r_0/z_0$ in Eq. (B4), we obtain

$$I(z_0) = \int_0^\infty \frac{\exp[-(1-i)k_s z(1+x^2)^{1/2}]}{1+x^2}\, x\mathrm{d}x$$

and, according to Eq. (B12) (see also Ref. 22, p. 134, entry 6),

$$I(z_0) = -\int_{(1+i)k_s z_0}^\infty \frac{e^{iy}}{y}\mathrm{d}y = -Ei[i(1+i)k_s z_0]$$

$$= E_1[(1-i)k_s z_0]. \tag{B13}$$

Equation (B13) is a convenient representation of the exponential integral function (see Ref. 23, p. 229, entry 5.1.11). For computational purposes the following series expansion is useful:

$$E_1(z) = -\gamma - \ln z - \sum_{n=1}^\infty \frac{(-1)^n z^n}{nn!}, \qquad |\arg(z)| < \pi,$$

$$\gamma = 0.5772156649 \qquad \text{(Euler's constant)}. \tag{B14}$$

The detector signal will thus have the form

$$S_{\text{detector}}(z_0) = \frac{\exp(i\pi/4)}{4i}(k_s z_0)E_1[(1-i)k_s z_0]. \tag{B15}$$

For large values of $k_s z_0$, i.e., for thermally thick solids consistent with assumption (46), the asymptotic expansion of E_1 prevails[23]:

$$E_1(z) \sim \frac{e^{-z}}{z}, \tag{B16}$$

and the detector signal can be simplified to

$$S_{\text{detector}}(z_0 \gg k_s^{-1}) \approx \frac{1}{4\sqrt{2}}\exp[-(1-i)k_s z_0], \tag{B17}$$

or, if we take the real part,

$$S_{\text{detector}}(z_0 \gg k_s^{-1}) \approx \frac{1}{4\sqrt{2}}\exp(-k_s z_0)\cos(k_s z_0). \tag{B18}$$

Relation (B18) is, in fact, the one-dimensional solution to the thermal-wave problem. It is valid for a semi-infinite solid[4] and gives the value of the harmonic temperature field at depth z_0. In physical terms, this argument shows that, far from the sample surface, the average thermal-wave field reduces to one-dimensional behavior with a total loss of transverse resolution.

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