Theory of combined acousto-photo-thermal spectral decomposition in condensed phases: parametric generation of thermal waves by a non-stationary ("breathing") sub-surface defect

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Abstract

The theoretical description of the parametric transformation of the thermal-wave frequency spectrum by a "breathing" (i.e. non-stationary) defect in the laser-induced heat flux is presented. We derive analytical descriptions of the thermal-wave spectrum in the cases of harmonic and rectangular-wave periodic modulation of the thermal resistance of the defect by external acoustic (or other mechanical) loading. Furthermore, we establish the conditions under which the proposed method of active acousto-photo-thermal diagnostics allows not only the detection of an otherwise photothermally invisible crack without acoustic loading, but also yields information on its depth beneath the surface.

1. Introduction

In the development of the modern physics of photothermal phenomena, significant efforts have been directed to the investigation of various possibilities of contactless diagnostics of sub-surface cracks and delaminations of thin coatings [1–4]. This impetus can be primarily explained by the practical importance of the non-destructive detection of these kinds of defects for applications [4]. In the simplest physical situations, both open cracks (gas-filled gaps) and closed cracks (intimate contact of the two surfaces of the crack) can be characterized by their thermal resistance R [1–6]. The possibility of the photothermal visualization of a "hidden" crack is determined by the sensitivity of the experimental set-up, the depth of the crack beneath the surface and the magnitude of the thermal resistance as well [4,6].

In the present work the photothermal response of a non-stationary defect under conditions of active (externally forced) variation of its thermal resistance is theoretically investigated. The main motivation of the proposed method for materials evaluation is the following simple and self-evident idea: if it is difficult to characterize a sub-surface crack by traditional photothermal methods (e.g. as a consequence of its small thermal resistance), then it is necessary to increase its thermal resistance artificially. This can be achieved by acoustical (mechanical) loading of the sample, as the induced elastic stresses may cause the enlargement of the crack (the increase of the gas-layer width). Under these conditions the thermal resistance of the thermally-thin gas layer, which is isolated from the external ambient, is directly proportional to its width h in both regimes of diffusional and ballistic thermal conductivity.

In fact, as the gas molecule mean free path l varies inversely proportionally to the gas pressure P, while P in the isolated crack is inversely proportional to h, then $l \propto h$. Consequently, there will be no transition in the gas heat-conduction regime from diffusional to ballistic (or vice versa) with increasing gas-layer thickness. If the thermal conductivity in the gas is diffusional ($l \ll h$) then the thermal resistance of the thermally-thin gas layer is given by the formula $R = h/K_g$, where K_g is the thermal conductivity of a dilute gas, and is independent of P and, consequently, of h [6,7]. In the case of ballistic heat conductivity ($l \gg h$) the thermal resistance is inversely proportional to P [5,7] and, since $P \sim 1/h$, it can be also expressed in the form $R = h/K_g^*$, where the magnitude of the h-independent quantity K_g^* is less that of K_g. Thus, if acoustic loading leads to an increase of the gas-layer thickness, then one may expect that the corresponding increase in the crack thermal resistance will make it easier to detect by photothermal means.

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Note that the proposed method of non-destructive evaluation has one important peculiarity. This method is only sensitive to the thermal resistances which can be modulated by external stresses, and not to the other thermal inhomogeneities of the sample (for example, solid inclusions composed of different material).

For experimental purposes, as well as for photothermal measurements, it is important to implement acoustic loading in a contactless manner by laser generation of sound in the sample [6,8].

It is, of course, desirable to find means to enhance the amplitude of the laser-launched ultrasound in the region of an existing crack, which, for our purposes, is the photothermally probed region. This can be achieved, for example, by use of the converging surface-acoustic-wave technique [9]. Furthermore, an alternative method to enhance the defect-related signal characteristics is through an increase of the sub-surface gas-layer thickness. This can be achieved experimentally by non-contact laser thermal loading at low frequencies [10]. Consequently, one can design an experimental situation where one (pump-)laser-induced thermal wave is used as the driving force to modulate the thermal resistance through thermoelastic stresses, while another interrogates the modulation ("breathing") of the thermal resistance. An essential aspect of this type of configuration is the investigation of the non-linear (or self-induced) photothermal processes occuring locally, i.e. the influence of the crack "breathing" on the laser-launched thermal wave, which itself is the source of the "breathing" phenomenon [11,12].

2. General mathematical formalism

To demonstrate the main features of the theoretical method of active combination of acousto-photo-thermal diagnostics of materials, consider the following geometry: laser radiation incident on the plane surface of a semi-infinite sample ($z \geqslant 0$) induces the heat flux $J_L(\boldsymbol{r},0;t)$ across the boundary $z=0$ (here $\boldsymbol{r}$ is a vector in the plane perpendicular to the z axis). Now consider a sub-surface defect (crack) localized at the depth $z=H$ parallel to the irradiated surface. The parameters of the defect are independent of $\boldsymbol{r}$, but can be modulated in time (non-stationary). Therefore, let us characterize the defect by its thermal resistance $R=R(t)$. The mathematical description of the laser-induced temperature variations T is given by the homogeneous equation of heat conduction with the corresponding boundary conditions [6]:

$$\left(\nabla^2-\frac{1}{D}\frac{\partial}{\partial t}\right)T(\boldsymbol{r},z;t)=0 \qquad (1a)$$

$$-k\frac{\partial}{\partial z}T(\boldsymbol{r},0;t)=J_L(\boldsymbol{r},0;t), \quad T(\boldsymbol{r},\infty;t)=0 \qquad (1b)$$

$$T(\boldsymbol{r},H+0;t)-T(\boldsymbol{r},H-0;t)$$
$$=Rk\frac{\partial}{\partial z}T(\boldsymbol{r},z=H;t)\equiv -R\Phi(\boldsymbol{r},t) \qquad (1c)$$

where D and k are the thermal diffusivity and the thermal conductivity of the sample, respectively, and we introduced the notation $\Phi(\boldsymbol{r},t)$ for the heat flux through the thermal resistance:

$$\Phi(\boldsymbol{r},t)\equiv -k\frac{\partial}{\partial z}T(\boldsymbol{r},z=H-0;t)$$
$$=-k\frac{\partial}{\partial z}T(\boldsymbol{r},z=H+0;t)$$

To fix our ideas, and for simplicity, let us consider that the laser beam has cylindrical symmetry. Then the application of the Fourier transform in time and of the Bessel transform in radial coordinate ($r=|\boldsymbol{r}|$) is appropriate:

$$F(t)=\int_{-\infty}^{\infty}\exp(-\mathrm{i}\omega t)\tilde{F}(\omega)\,\mathrm{d}\omega \qquad (2a)$$

$$\tilde{F}(\omega)=(2\pi)^{-1}\int_{-\infty}^{\infty}\exp(\mathrm{i}\omega t)F(t)\,\mathrm{d}t \qquad (2b)$$

$$F(r)=\int_0^{\infty}J_0(qr)\hat{F}(q)\,q\,\mathrm{d}q \qquad (2c)$$

$$\hat{F}(q)=\int_0^{\infty}J_0(qr)F(r)\,r\,\mathrm{d}r \qquad (2d)$$

Here F is an arbitrary function and J_0 is the Bessel function of the zeroth order. This set of transformations reduces the partial differential problem of Eq. (1) to the analysis of an ordinary Fourier–Bessel transform differential equation in the z variable. The solution of this equation satisfying the boundary requirements at $z=0$ and $z=\infty$ is:

$$\hat{\tilde{T}}(0\leqslant z\leqslant H-0)=\hat{\tilde{T}}(0)\cosh(pz)$$
$$-\frac{1}{kp}\hat{\tilde{J}}_L\sinh(pz) \qquad (3)$$

$$\hat{\tilde{T}}(z\geqslant H+0)=\frac{1}{kp}\hat{\tilde{\Phi}}\exp[-p(z-h)] \qquad (4)$$

where $p(q,\omega)\equiv\sqrt{q^2-\mathrm{i}\omega/D}$, $Rep(q,\omega)>0$ and $\hat{\tilde{T}}(0)$ is the combined Fourier–Bessel transform of the temperature of the irradiated surface ($z=0$). $\hat{\tilde{\Phi}}(q,\omega)$ is the Fourier–Bessel transform of $\Phi(\boldsymbol{r},t)$.

The first relation, between the coefficients $\hat{\hat{T}}(0)$ and $\hat{\hat{\Phi}}$ follows from Eq. (3) and the definition of Φ:

$$\hat{\hat{\Phi}} \equiv -k\frac{\partial}{\partial z}\hat{\hat{T}}(r,z=H-0;t)$$

By differentiation of Eq. (3) we find

$$\hat{\hat{\Phi}} = -kp\hat{\hat{T}}(0)\sinh(pH)+\hat{\hat{J}}_{\rm L}\cosh(pH) \tag{5}$$

The second relation can be obtained by substituting Eqs. (3) and (4) in the boundary condition at $z=H$, Eq. (1c):

$$\frac{1}{kp}\hat{\hat{\Phi}} - \hat{\hat{T}}(0)\cosh(pH)+\frac{1}{kp}\hat{\hat{J}}_{\rm L}\sinh(pH) = -(\widetilde{R\hat{\hat{\Phi}}}) \tag{6}$$

It can be seen that the expression $(\widetilde{R\hat{\hat{\Phi}}})$ on the right-hand side of Eq. (6) cannot be deconvolved because both functions R and $\hat{\hat{\Phi}}$ are time-dependent.

It is convenient to eliminate $\hat{\hat{\Phi}}$ from the left-hand side of Eqs. (5) and (6) to obtain the following expression for the surface temperature:

$$\hat{\hat{T}}(0)=\frac{1}{kp}\hat{\hat{J}}_{\rm L}+\exp(-pH)(\widetilde{R\hat{\hat{\Phi}}}) \tag{7}$$

Eq. (7) connects the surface temperature (usually detected in the photothermal experiments) with the flux across the thermal resistance. The physical meaning of Eq. (7) is clear: the first term on the right-hand side describes the surface temperature in the absence of thermal resistance ($R\equiv 0$), a simple dependence on the surface thermal flux. The second term describes the temperature changes caused by the reflection of the thermal waves at the boundary $z=H$. The exponential factor $\exp(-pH)$ determines the attenuation of thermal waves in their propagation from the thermal resistance ($z=H$) to the probed surface ($z=0$).

Eliminating $\hat{\hat{T}}(0)$ from Eqs. (5) and (6), we derive the closed-form equation for the Fourier–Bessel transform of the heat flux $\Phi(\boldsymbol{r},t)$

$$\hat{\hat{\Phi}} = \hat{\hat{J}}_{\rm L}\exp(-pH) - kp\sinh(pH)\exp(-pH)(\widetilde{R\hat{\hat{\Phi}}}) \tag{8}$$

The first term on the right-hand side of Eq. (8) describes the laser-induced heat flux at the depth $z=H$ in the absence of thermal resistance. The second term accounts for multiple thermal-wave reflections between the thermal resistance and the irradiated surface.

Eq. (8), obtained as a result of the assumed high symmetry of the investigated physical model, seems to be one of the simplest equations suitable for the analysis of combined acousto-photo-thermal effects. In the general case

$$(\widetilde{R\hat{\hat{\Phi}}}) = \int_{-\infty}^{\infty} d\omega'\tilde{R}(\omega')\hat{\hat{\Phi}}(q,\omega-\omega') \tag{9}$$

Therefore, Eq. (8) is the integral equation for the frequency spectrum of the heat flux through the thermal resistance. In the case of periodic modulation of the thermal resistance,

$$R(t) = \sum_{N=-\infty}^{\infty} R_n\, e^{-in\omega_0 t}$$

$$R_n = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} R(t)\, e^{-in\omega_0 t}\, dt \tag{10}$$

Here τ is the period of modulation and $\omega_0=2\pi/\tau$ is the angular frequency. Under these conditions, Eq. (8) becomes a functional equation [13], since Eqs. (9) and (10) yield

$$(\widetilde{R\hat{\hat{\Phi}}}) = \sum_{n=-\infty}^{\infty} R_n\,\hat{\hat{\Phi}}(\omega-n\omega^0) \tag{11}$$

Furthermore if $J_{\rm L}$ is also periodically modulated, then Eq. (8) reduces to a difference equation [14].

In all these cases the possibility of analytical solution of Eq. (8) looks questionable because of the complicated frequency dependence of the coefficients in Eq. (8).

We have managed to solve the problem of Eqs. (7) and (8) in the case when the thermal resistance is thermally localized close to the irradiated surface, i.e.

$$|p|H \ll 1 \tag{12}$$

which simplifies coefficients in Eq. (8). Before starting this analysis, let us examine the solution of Eqs. (7) and (8) in the limit of extremely small thermal resistance

$$k|p|R \ll 1 \tag{13}$$

Using the stepwise successive approximation method [15] we obtain as the second step

$$\hat{\hat{T}}(0) \approx \frac{1}{kp}\hat{\hat{J}}_{\rm L}+\exp[-p(q,\omega)H] \times \sum_{n=-\infty}^{\infty} R_n\hat{\hat{J}}_{\rm L}(\omega-n\omega_0)\exp[-p(q,\omega-n\omega_0)H] \tag{14}$$

For concreteness, $R(t)$ is now taken to be a periodic function, Eq. (10). In agreement with Eq. (14), the variation of the thermal resistance with time causes the parametric transformation of the laser-induced heat-flux modulation spectrum. It is important to note that in the spectrum of the surface temperature there may appear new frequencies, which are absent in the absence of thermal resistance modulation. For example, in the case of harmonic modulation of the laser intensity, $J_L(t) \sim [1+\sin(\omega_L t)]$, *and* harmonic modulation of the thermal resistance, $R(t) \sim [1+\sin(\omega_0 t)]$, the additional frequencies ω_0, $\omega_L - \omega_0$ and $\omega_L + \omega_0$ appear in the spectrum of the surface temperature already at the second step, Eq. (14), of successive approximations, (i.e. after the first reflection of thermal waves from the thermal resistance). Note that progressive broadening of the thermal-wave spectrum with the order of reflection from the thermal resistance also occurs in the absence of laser intensity modulation.

These simple considerations illustrate one of the practical advantages of the combined acousto-photo-thermal diagnostics, that is: acoustic modulation of the thermal resistance may lead to the parametric generation of new components in the surface temperature spectrum. Detection of these new spectral components may significantly increase the sensitivity of photothermal characterization of non-stationary cracks, i.e. those which are able to "breathe". Such a photothermal detection mode has the distinct advantage of background (fundamental frequency) signal suppression via the monitoring of harmonics with zero background which arise only if a sub-surface defect exists, i.e. signal selectivity of defects superior to the conventional photothermal imaging of near-surface defects.

3. Thermally-thin overlayer

Let us now analyze the situation of a thermal resistance thermally localized close to the sample surface, according to relation (12). Notice that this situation is typical of some experimental configurations, such as in Refs. [1,2]. Under the condition (12), Eq. (8) simplifies to

$$\hat{\hat{\Phi}} \approx \hat{\hat{J}}_L - kH\left(q^2 - \mathrm{i}\frac{\omega}{D}\right)\widetilde{R\hat{\Phi}} \quad (15)$$

Consequently, the inverse Fourier transformation (Eq. (2a)) leads to the differential equation for the function $\hat{\Phi}(q,t)$:

$$\hat{\Phi} \approx \hat{J}_L - kH\left(q^2 + \frac{1}{D}\frac{\partial}{\partial t}\right)R\hat{\Phi} = 0 \quad (16)$$

This equation can be conveniently rewritten by introducing the characteristic times

$$\tau_q \equiv 1/(Dq^2) \qquad \tau_R \equiv \mathrm{CHR} \quad (17)$$

where $C = k/D$ is the heat capacity per unit volume:

$$\frac{\partial}{\partial t}(R\hat{\Phi}) + \left[\frac{1}{\tau_q} + \frac{1}{\tau_R(t)}\right](R\hat{\Phi}) = \frac{1}{CH}\hat{J}_L \quad (18)$$

Under the condition (12), Eq. (7) gives the following description of the temperature variation Fourier-Bessel transform, $\Delta\hat{\hat{T}}$, related to the existence of a thermal resistance

$$\Delta\hat{\hat{T}} \equiv \hat{\hat{T}}(0) - \frac{1}{kp}\hat{\hat{J}}_L \approx \widetilde{R\hat{\Phi}} \quad (19)$$

Combining Eqs. (18) and (19), we arrive at the equation

$$\frac{\partial}{\partial t}\Delta\hat{T} + \left[\frac{1}{\tau_q} + \frac{1}{\tau_R(t)}\right]\Delta\hat{T} = \frac{1}{CH}\hat{J}_L \quad (20)$$

From the physical point of view τ_q is the characteristic diffusional cooling time of a strip of thickness on the order of $1/q$. When the laser beam is focused on a gaussian spot with a radius r_0, then the q spectrum of the launched photothermal waves has an upper bound: $q \lesssim 2/r_0$. Consequently, in Eq. (20) $\tau_q \gtrsim r_0^2/4D \equiv \tau_r$. The introduced time τ_R is the characteristic time of diffusional heat transfer from a thin homogeneously heated sub-surface strip of thickness on the order of H, as a result of heat transfer through the thermal resistance. In accordance with Eq. (20) it is the acoustic loading of the sample which causes the modulation of the characteristic time τ_R.

In due consideration of the classical theory of heat transfer, in the model of Eqs. (18) and (19) (or, equivalently, Eq. (20)), we proceed to treat the sub-surface layer of thickness H as a lumped heat-capacity system [16] of uniform temperature. Then τ_R is called the *time constant* of the layer [16]. A reasonably uniform temperature distribution is expected in the sub-surface layer, if its internal resistance to heat transfer by conduction is small compared with the externally modulated thermal resistance of the crack

$$H/k \ll R \quad (21)$$

Under the condition (21), the major temperature gradient would occur throughout the crack itself (i.e. the gas-filled layer). Let us note that if the inequality (21) does not hold (i.e. $H/k \gtrsim R$), then for the thermally-thin layer, condition (12), the inequality $k|p|R \ll 1$ will be valid. Therefore, in agreement with condition (13), the problem may be solved by the stepwise successive approximation method [15].

The solution of Eq. (20) satisfying the initial condition $\Delta\hat{T}(t \to -\infty) = 0$ is

$$\Delta\hat{T}(t) = \frac{1}{CH}\int_0^\infty \mathrm{d}t'\,\hat{J}_L(t-t')\exp\left\{-\frac{t'}{\tau_q} - \int_{t-t'}^{t}\frac{\mathrm{d}t''}{\tau_R(t'')}\right\} \quad (22)$$

In the case of a gaussian laser intensity distribution, i.e. $J_L = J_L(t)\exp(-r^2/r_0^2)$, the inverse Bessel transformation, Eq. (2c), of Eq. (22) leads to the following description/solution of the surface-temperature variations along the axis of the system ($r=0$):

$$\Delta T(0,0;t)=\frac{1}{CH}\int_0^{\infty} \mathrm{d}t' \frac{J_L(t-t')}{1+t'/\tau_r} \exp\left\{-\int_{t-t'}^{t} \frac{\mathrm{d}t''}{\tau_R(t'')}\right\} \tag{23}$$

We will use Eq. (23) in our further analysis.

As two important examples demonstrating the nature and power of the mathematical approach developed, we will examine harmonically modulated thermal resistance R

$$R(t)=R_0[1+\sin(\omega_0 t)] \tag{24}$$

and rectangular-periodic modulation

$$R(t)=\begin{cases} 0, & \text{for } \left(\dfrac{2m-1}{2}\right)\tau<t<m\tau \\ R_0, & \text{for } m\tau<t<\left(\dfrac{2m+1}{2}\right)\tau \end{cases} \tag{25a}$$

$m=0, \pm 1, \pm 2, \ldots$

$$=R_0 \sum_{n=-\infty}^{\infty} \rho_n \, e^{-in\omega_0 t}$$

$$=R_0 \left\{\frac{1}{2}+\sum_{n=1}^{\infty} \frac{2}{\pi(2n-1)} \sin[(2n-1)\omega_0 t]\right\} \tag{25b}$$

Here $R_0=$ const. and

$$\rho_n=\frac{\exp(i\pi n)-1}{2i\pi n} \tag{25c}$$

are the dimensionless complex amplitudes of the spectral components of the modulation function. The modulation Eq. (24) introduces into the system only non-zero average resistance and the fundamental frequency ω_0, while the spectrum of the modulation, Eq. (25a), is wide-band, albeit discrete. Note that the square-wave spectral decomposition function, Eq. (25b), contains only odd harmonics of the fundamental frequency.

In agreement with the adopted models, Eqs. (24) and (25), the thermal resistance periodically becomes equal to zero. Thus, we will mainly turn to the examination of the situation when the magnitude of the thermal resistance actively induced and modulated by external acoustic loading considerably exceeds the value of possible residual thermal resistance of the closed crack. Lastly, for the sake of completeness we will investigate in this work the case of c.w. laser action, i.e. unmodulated input thermal flux to the sample surface in the presence of a.c. acoustic modulation of the defect. In other words, we will examine the process of parametric thermal-wave excitation by an acoustically driven breathing crack alone, when only a constant heat flux on the system is present in the absence of acoustic loading.

4. One-dimensional geometry

One of the important features of the physical phenomena under the present investigation is the existence of limited convergent solutions for ΔT even in the one-dimensional (1-D) approximation (i.e. $\partial J_L/\partial r\equiv 0$), when the background temperature T itself exhibits an unbounded increase near the surface: $T(0;t)\sim\sqrt{t}$. The physical explanation of this follows from the fact that in agreement with Eq. (19) ΔT is caused by the modulation of the heat flux, which is bounded, but not by the temperature field, which diverges.

4.1. Harmonic thermal-resistance modulation

In the 1-D geometry ($\tau_r \gg \tau$, τ_R, i.e. $r_0\to\infty$) typical of some photothermal depth-profiling experiments, the temperature variation, Eq. (23), induced by the harmonic modulation of the thermal resistance, Eq. (24), can be expressed in the form:

$$\Delta T(0,0;t)=2\theta_{\omega_0}\left\{f\left(\frac{2}{\omega_0\tau_R}\right)+\sum_{n=1}^{\infty}\Gamma(n)\mathrm{Im}\left[U\left(n,0,i\frac{2}{\omega_0\tau_R}\right)\times\exp\left(-in\omega_0 t-i\frac{\pi}{2}n\right)\right]\right\} \tag{26}$$

where $\theta_{\omega 0}\equiv J_L/cH\omega_0$ is the characteristic amplitude of thermal waves, $\Gamma(n)$ is the gamma function, U is the confluent hypergeometric function and f is the auxiliary function related to sine and cosine integrals Si and Ci [17]

$$f(x)=\mathrm{Ci}(x)\sin(x)-\left[\mathrm{Si}(x)-\frac{\pi}{2}\right]\cos(x) \tag{27}$$

In the limit of weak modulation of the thermal resistance when the characteristic time τ_R is much less than the period of the resistance modulation ($\tau_R\omega_0\ll 1$) the solution Eq. (26) takes the form:

$$\Delta T(0,0;t)=\theta_{\omega_0}\left\{\tau_R\omega_0+\sum_{n=1}^{\infty}\frac{\Gamma(n)}{2^{n-1}}(\tau_R\omega_0)^n\times\sin[n\omega_0 t-\pi(n-1)]\right\} \tag{28}$$

Eq. (28) explicitly describes the broadening of the thermal-wave spectrum as a result of multiple reflections of thermal waves from the time-modulated thermal resistance. From the structure of Eq. (28) it can be seen that the nth harmonic of the fundamental frequency is excited after the nth reflection of the thermal flux from the resistance R_0. Its dimensionless amplitude A_n is proportional to $(\tau_R\omega_0)^n \sim R_0^n$ with a phase delay equal to $\phi_n = \pi(n-1)$. Thus, all the odd harmonics start to grow with increasing R_0 in phase with the fundamental frequency, while all the even harmonics grow out of phase with the fundamental.

In the limit of strong modulation of the thermal resistance ($\tau_R\omega_0 \gg 1$) the solution Eq. (26) transforms to:

$$\Delta T(0,0;t) \approx \theta_{\omega_0}\left\{\pi + \sum_{n=1}^{\infty}\frac{2}{n} \times \sin\left[n\omega_0 t - \frac{\pi}{2}(3n-2)\right]\right\} \tag{29}$$

Eq. (29) describes the saturation of both the contribution to the average temperature and the amplitudes of the harmonics ($A_n \sim 1/n$). It shows that the phase of the nth harmonic for $R_0 \to \infty$ exhibits an additional delay equal to $(\pi/2)n$ relative to its phase for $R_0 \to 0$.

In order to describe the evolution of the spectrum of the parametrically excited thermal waves with increasing thermal resistance in the intermediate regime, it is necessary to use the exact analytical solution, Eq. (26). Towards this goal, note that the calculation of $U(n,0,z)$ for $n \geqslant 3$ can be performed with the help of the recurrence relation [17]

$$U(n,0,z) = \frac{1}{n(n-1)} \times \{[2(n-1)-z]U(n-1,0,z) - U(n-2,0,z)\} \tag{30}$$

Therefore, all calculations of the confluent hypergeometric function are reduced to the calculations of $U(2,0,z)$ and $U(1,0,z)$, which are also related:

$$U(2,0,z) = \tfrac{1}{2}[(1+z)U(1,0,z) - zU(1,1,z)] \tag{31}$$

Furthermore, the functions $U(1,1,z)$ and $U(1,0,z)$ can be expressed through the auxiliary functions f and g related to sine and cosine integrals [17]

$$U(1,1,\mathrm{i}x) = g(x) - \mathrm{i}f(x) \tag{32}$$

$$U(1,0,\mathrm{i}x) = 1 - xf(x) - \mathrm{i}xg(x) \tag{33}$$

where $f(x)$ was defined in Eq. (27) and $g(x)$ is given by

$$g(x) = -\mathrm{Ci}(x)\cos(x) - \left[\mathrm{Si}(x) - \frac{\pi}{2}\right]\sin(x) \tag{34}$$

Then, in view of Eqs. (30) and (31), the description of any thermal-wave harmonic can be given in terms of the functions f and g.

The results of the calculations of the dimensionless amplitudes and phases of the three first harmonics and their contribution to the average temperature (A_0) as functions of the dimensionless parameter $\tau_R/\tau = (\tau_R\omega_0)/2\pi$ are presented in Figs. 1 and 2. These

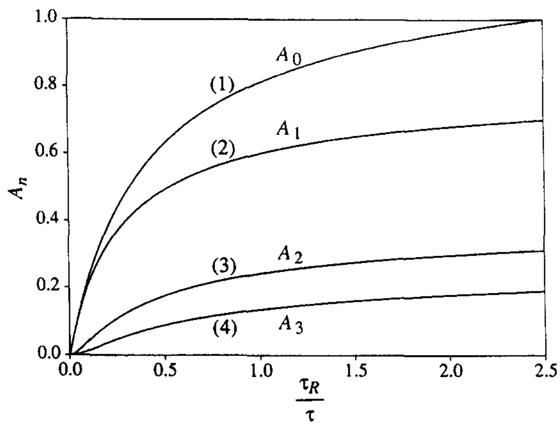

Fig. 1. Dependence of the dimensionless amplitudes of the spectral decomposition of the surface temperature, Eq. (28), on the parameter (τ_R/τ) in the case of harmonic modulation of the thermal resistance of a non-stationary defect (crack) in the 1-D regime. Curve (1), A_0 contribution to the average temperature; curves (2)–(4), contribution of the thermal-wave harmonics A_1–A_3.

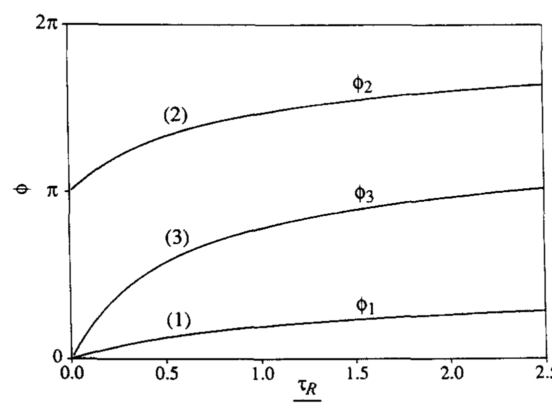

Fig. 2. Dependence of the phases of the spectral decomposition of the surface temperature, Eq. (29), on the dimensionless parameter (τ_R/τ) in the case of harmonic modulation of the thermal resistance of a non-stationary defect (crack) in the 1-D regime. Curves (1)–(3), contributions of the phases ϕ_1–ϕ_3 of the thermal-wave harmonics.

plots of Eq. (26) exhibit the asymptotic behavior indicated by Eqs. (28) and (29) in the limits of small and large values of τ_R/τ, respectively.

4.2. Rectangular-wave thermal-resistance modulation

In the case of rectangular periodic modulation of the thermal resistance, Eq. (25), the solution Eq. (23) may be expressed in terms of special functions without the one-dimensionality assumption:

$$\Delta t(0,0;t) = \frac{J_L \tau_r}{CH} \begin{cases} 0, \text{ for } \left(\frac{2m-1}{2}\right)\tau < t < m\tau; \\ \exp\left(\frac{\tau_r}{\tau_R}\right)\left[E_1\left(\frac{\tau_r}{\tau_R}\right) - E_1\left(\frac{\tau_r + t}{\tau_R}\right)\right] \\ \text{for } m\tau < t < \left(\frac{2m+1}{2}\right)\tau \end{cases} \tag{35}$$

where $m = 0, \pm 1, \pm 2, \ldots$, and E_1 is the exponential integral [17]. The expansion of Eq. (35) in Fourier series with complex amplitudes leads to the following expression for the nth Fourier component of $\Delta T(0,0;t)$:

$$\Delta T_n = \frac{J_L \tau_r}{CH} \exp\left(\frac{\tau_r}{\tau_R}\right)\left[\rho_n E_1\left(\frac{\tau_r}{\tau_R}\right) - \frac{1}{2\pi i n}\right.$$
$$\times \left(\exp(i\pi n) E_1\left(\frac{\tau_r}{\tau_R} + \frac{\tau}{2\tau_R}\right) - E_1\left(\frac{\tau_r}{\tau_R}\right)\right.$$
$$+ \exp\left(2\pi i n \frac{\tau_r}{\tau}\right)\left\{E_1\left(\frac{\tau_r}{\tau_R} - 2\pi i n \frac{\tau_r}{\tau}\right)\right.$$
$$\left.\left.\left. - E_1\left[\left(\frac{\tau_r}{\tau_R} - 2\pi i n \frac{\tau_r}{\tau}\right)\left(1 + \frac{\tau}{2\tau_r}\right)\right]\right\}\right)\right] \tag{36}$$

ρ_n was defined in Eq. (25c). The derived expression Eq. (36) completes the description of the signal after insertion in a conventional complex Fourier series expansion of $\Delta T(0,0;t)$, Eq. (25b).

In the 1-D geometry ($\tau_r \gg \tau, \tau_R$) the general solution Eq. (36) reduces to:

$$\Delta T_n = \frac{J_L \tau_r}{CH}\left\{\rho_n - \frac{1 - \exp[i\pi n - (\tau/2\tau_R)]}{(\tau/\tau_R) - 2\pi i n}\right\} \tag{37}$$

Using the expression for ρ_n, Eq. (25c), it is convenient to rewrite Eq. (37) as

$$\Delta T_n = \theta_\tau \left\{\rho_n - \frac{\tau_R}{\tau}\exp(i\pi n)\left[1 - \exp\left(-\frac{\tau}{2\tau_R}\right)\right]\right\} \tag{38}$$

where $\theta_\tau \equiv J_L \tau / CH$ is the characteristic amplitude of thermal waves. To compare the spectrum of the thermal waves to that of the resistance modulation it is useful to present the temperature field ΔT in the form

$$\Delta T(0,0;t) = \theta_\tau \left[A_0 + \sum_{n=1}^{\infty} A_n \sin(n\omega_0 t - \phi_n)\right] \tag{39}$$

where

$$A_0 = \frac{\tau_R}{\tau}\left\{\frac{1}{2} - \frac{\tau_R}{\tau}\left[1 - \exp\left(-\frac{\tau}{2\tau_R}\right)\right]\right\} \tag{40a}$$

$$A_n = \frac{2}{\sqrt{(\tau/\tau_R)^2 + (2\pi n)^2}} \times \sqrt{\left[\frac{1-(-1)^n}{2\pi n}\right]^2 + \left\{\frac{\tau_R}{\tau}\left[1 - \exp\left(-\frac{\tau}{2\tau_R}\right)\right]\right\}^2} \tag{40b}$$

$$\phi_n = \tan^{-1}\left(2\pi n \frac{\tau_R}{\tau}\right) - \tan^{-1}\left\{\pi n \frac{\tau_R}{\tau}\left[1 - \exp\left(-\frac{\tau}{2\tau_R}\right)\right]\right\}$$
$$\text{for } n \text{ odd } (n = 2m-1,\ m = 1, 2, \ldots) \tag{40c}$$

$$\phi_n = \frac{\pi}{2} \tan^{-1}\left(2\pi n \frac{\tau_R}{\tau}\right)$$
$$\text{for } n \text{ even } (n = 2m,\ m = 1, 2, \ldots) \tag{40d}$$

In agreement with Eqs. (40a)-(40d) in the limit of weak thermal resistance modulation ($\tau_R \ll \tau/2\pi n$)

$$A_0 \approx (1/2)(\tau_R/\tau) \tag{41a}$$

$$A_{2k-1} \approx (2/\pi n)(\tau_R/\tau) \tag{41b}$$

$$A_{2k} \approx 2(\tau_R/\tau)^2 \tag{41c}$$

Consequently, all the even harmonics of the fundamental frequency are seen to be excited already after the second reflection of the thermal flux from the nonstationary crack. This is one of the major differences between the cases of narrow-band (Eq. (24)) and wide-band (Eq. (25b)) thermal-resistance modulation. For small (τ_R/τ) the odd spectral harmonics are synchronous with the thermal resistance modulation: $\phi_{2k-1} \approx \pi n(\tau_R/\tau) \ll 1$ (Eq. (40c)), while the even harmonics are excited with a phase delay of $\pi/2$: $\phi_{2k} \approx \pi/2 + 2\pi n(\tau_R/\tau)$ (Eq. 40d)). However, in the limit of strong thermal resistance modulation ($\tau_R \gg \tau$) the amplitudes of all the components saturate:

$$A_0 \approx 1/8 \tag{42a}$$

$$A_n \approx \frac{1}{2\pi n}\sqrt{1 + \left[\frac{1-(-1)^n}{\pi n}\right]^2} \tag{42b}$$

The phases also saturate for $\tau_R \gg \tau/2\pi n$:

$$\phi_{2k-1} \approx \tan^{-1}[2/\pi(2k-1)] \tag{43a}$$

$$\phi_{2k} \approx \pi \tag{43b}$$

For sufficiently large numbers n, one can further examine the behavior of the thermal waves generated under the condition $\tau/2\pi n \ll \tau_R \ll \tau$. If these inequalities are valid, then the amplitudes of both odd and even harmonics are increasing proportionally to R_0:

$$A_n \approx \frac{\tau_R}{\tau}\sqrt{1+\left[\frac{1-(-1)^n}{\pi n}\right]^2} \tag{44}$$

For $n \gg 1$ and $\tau_R \ll \tau$, Eq. (40c) for the phases of the odd harmonics reduces to the form

$$\phi_n \approx \tan^{-1}\left[\frac{\pi n(\tau_R/\tau)}{1+2(\pi n \tau_R/\tau)^2}\right] \tag{45}$$

which exhibits non-monotonic dependence of the phase on τ_R/τ.

The results of our calculations of the dimensionless amplitudes A_0–A_4 and the phases ϕ_1–ϕ_5 as functions of the dimensionless parameter τ_R/τ are presented in Figs. 3 and 4. These plots are based on Eqs. (40a)–(40d). According to Fig. 4, the non-monotonic phase changes of the odd harmonics commence with $n=3$.

The theory developed in this work predicts a significant dependence of both the amplitudes and the phases of the thermal waves in 1-D geometry on the relative magnitudes of the time constant τ_R of the sample and the thermal resistance modulation period τ. The plots in Figs. 1–4 demonstrate that, observing experimentally the effects of the onset of amplitude or phase saturation, one can identify the situation when $\tau_R \approx \tau$. Thus it is possible to measure experimentally the quantity $HR_0 \approx \tau/C$ from the known value of the thermal resistance modulation period τ. If the thermal resistance can be determined independently, then this parametric thermal-wave technique provides a method of characterizing the position of the crack (i.e. the depth H) beneath the irradiated surface.

If one is interested in a self-consistent experimental method, then there is another possibility to determine thermal resistance. In the limit of weak thermal resistance modulation (i.e. $\tau_R/\tau \ll 1$) the dimensional amplitudes of those spectral components of $\Delta T(0,0;t)$ which initially exist in the spectrum of $R(t)$ do not depend on the depth H because they are proportional to $\theta_{\omega 0,\tau}(\tau_R/\tau) \sim J_L R_0$, where the symbol $\theta_{\omega 0,\tau}$ stands for either $\theta_{\omega 0}$ or θ_τ. Thus increasing the modulation period τ from $\tau_0 \sim \tau_R$ to $\tau_\infty \gg \tau_R$, we arrive at the possibility of experimentally determining the thermal resistance R_0 (and then the sub-surface crack position $H \approx \tau_0/CR_0$) by performing absolute-temperature measurements. We also note that, according to Eqs. (29), (40a) and (40b), in the limit of strong thermal resistance modulation ($\tau_R \gg \tau$) the thermal-wave amplitudes become independent of R_0 ($\Delta T \sim \theta_{\tau,\omega} \sim J_L\tau/CH$). Consequently, in the regime $\tau_R \gg \tau$ the problem of the determination of the overlayer thickness H reduces to the absolute measurement of the thermal-wave amplitude at the probing surface.

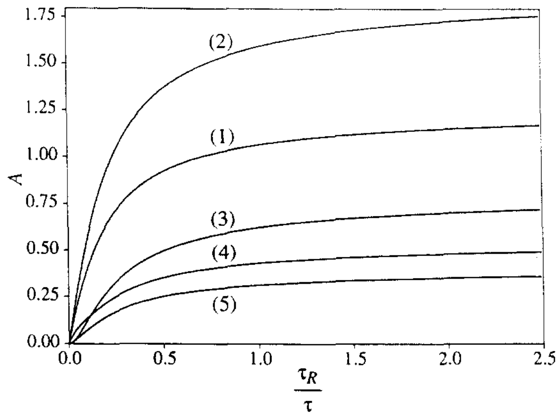

Fig. 3. Dependence of the amplitudes of the spectral decomposition of the surface temperature, Eqs. (40a) and (40b), on the parameter (τ_R/τ) in the case of rectangular-wave periodic modulation of the thermal resistance of a non-stationary defect (crack) in the 1-D regime. Curve (1), A_0 contribution to the average temperature; curves (2)–(5), contributions of the thermal-wave harmonics A_1–A_4.

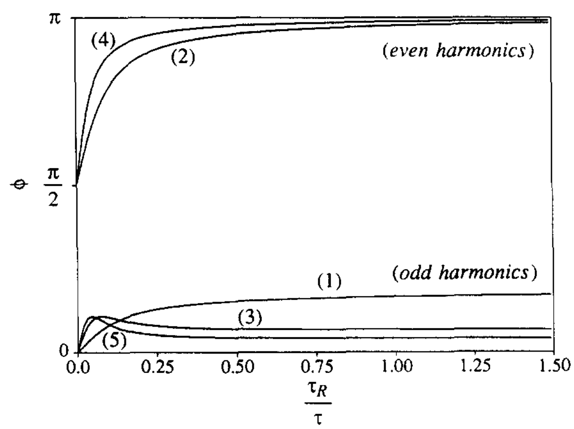

Fig. 4. Dependence of the phases of the spectral decomposition of the surface temperature, Eqs. (40c) and (40d), on the parameter (τ_R/τ) in the case of rectangular-wave periodic modulation of the thermal resistance of a non-stationary defect (crack) in the 1-D regime. Curves (1)–(5), contributions of the phases ϕ_1–ϕ_5 of the thermal-wave harmonics.

5. Quasi-stationary regime

In usual photothermal experiments the quasi-stationary regime is typical for photothermal microscopy. In this mode the penetration length of the thermal wave is controlled by 3-D heat diffusion from the laser-heated surface spot, i.e. $\tau \gg \tau_r$. In the combined acousto-photo-thermal configuration examined in the present work, the quasi-stationary condition is described by the inequality $\tau \gg (\tau_r, \tau_R)$. Under this condition one can proceed further in the general solution of the problem, omitting the time derivative in Eq. (20). Then the solution corresponding to Eq. (23) is:

$$\Delta T(0,0;t) = \frac{\tau_r}{CH} J_L(t) \exp[\tau_r/\tau_R(t)]\, E_1[\tau_r/\tau_R(t)] \tag{46}$$

In the 1-D geometry ($\tau_r \gg \tau_R$), Eq. (46) reduces to

$$\Delta T(0,0;t) \approx J_L(t) R(t) \tag{47}$$

i.e. to the case of 1-D quasi-stationary frequency mixing. Note that information on the overlayer thickness H completely disappears from Eq. (47).

5.1. Harmonic thermal-resistance modulation

In the case of harmonic modulation of the thermal resistance, Eq. (24), the solution Eq. (46) can be presented in the form:

$$\Delta T(0,0;t) = \theta_r \left\{ A_0 + \sum_{n=1}^{\infty} A_n \sin\left[n\omega_0 t - \frac{3\pi}{2}(n-1) \right] \right\} \tag{48}$$

where

$$A_0 = \int_0^{\infty} \frac{dx}{\sqrt{x+1}\,(\sqrt{x+1}+1)} \exp\left(-\frac{\tau_r}{2\tau_R} x \right) \tag{49a}$$

$$A_n = 2 \int_0^{\infty} \frac{x^{n-1}\, dx}{\sqrt{x+1}\,(\sqrt{x+1}+1)^{2n}} \exp\left(-\frac{\tau_r}{2\tau_R} x \right) \tag{49b}$$

Here $\theta_r \equiv J_L \tau_r / CH$ is the characteristic amplitude of temperature variations.

According to Eq. (48), each next higher harmonic in the quasi-stationary regime is delayed in phase relative to the one before, i.e. $\phi_{n+1} - \phi_n = 3\pi/2$. In the case of weak thermal resistance modulation ($\tau_R \ll \tau_r$), Eq. (48) leads to $A_0 \approx \tau_R/\tau_r$ and $A_n \approx 2\Gamma(n)(\tau_R/\tau_r)^n$, demonstrating that the nth harmonic is parametrically generated after the sequence of n reflections of thermal flux from the breathing defect. In the limit of strong modulation ($\tau_R \gg \tau_r$), the solution Eq. (48) describes the saturation of the harmonics ($A_n \approx 2/n$) and a logarithmic zeroth-mode contribution to the average temperature: $A_0 \approx \ln(\tau_R/\tau_r)$.

For the analysis of the intermediate regime, one must use the general solution, Eq. (48). It is convenient to present the amplitudes in the form

$$A_0 = \Psi(1) \tag{50a}$$

$$A_n = \sum_{k=0}^{n-1} (-1)^k 2^{k+1} \binom{n-1}{k} \Psi(k+2) \tag{50b}$$

where we introduced the binomial coefficients $\binom{n}{k}$ [17] and the notation $\Psi(m)$ for the integral

$$\Psi(m) \equiv \int_0^{\infty} \frac{dx}{\sqrt{x+1}\,(\sqrt{x+1}+1)^m} \exp\left(-\frac{\tau_r}{2\tau_R} x \right) \tag{51}$$

The recursion relation

$$\Psi(m \geq 2) = \frac{2}{m-1} \left\{ \frac{1}{2^{m-1}} + \frac{\tau_r}{2\tau_R} [\Psi(m-1) - \Psi(m-2)] \right\} \tag{52}$$

gives a way of reducing the calculation of the amplitudes to the known functions

$$\Psi(0) = \sqrt{2\pi \left(\frac{\tau_R}{\tau_r} \right)} \exp\left(\frac{\tau_r}{2\tau_R} \right) \operatorname{erfc}\left(\sqrt{\frac{\tau_r}{2\tau_R}} \right) \tag{53}$$

and

$$\Psi(1) = \Psi(0) - \int_0^{\infty} \frac{dx}{\sqrt{x+1}+1} \exp\left(-\frac{\tau_r}{2\tau_R} x \right) \tag{54}$$

The latter integral can be expressed in terms of the generalized hypergeometric functions ${}_2F_2$ [18]; however, it is more convenient to calculate it numerically. The results of the calculations of the dependence of the dimensionless amplitudes A_0–A_3 on the parameter τ_R/τ_r are presented in Fig. 5 (curves (1)–(4)).

5.2. Rectangular-wave thermal-resistance modulation

In the case of rectangular-wave periodic modulation of the thermal resistance, Eq. (25a), the solution Eq. (46) gives the following description for the nth-component complex amplitude of the photothermal waves:

$$\Delta T_n = \theta_r \exp\left(\frac{\tau_r}{\tau_R} \right) E_1\left(\frac{\tau_r}{\tau_R} \right) \rho_n \tag{55}$$

According to Eq. (55) the spectrum of $\Delta T(0,0;t)$ is identical to that of $R(t)$. We graphically present the dependence of the dimensionless normalized amplitude component ΔT_n of the thermal wave (i.e. $\Delta T_n/\theta_r\rho_n$) in Fig. 5 (curve (5)). The even harmonics of the fundamental frequency appear in the quasi-stationary regime only in the next approximation in the small parameter (τ_r/τ). Using the condition $\tau \gg (\tau_R, 2\pi n\tau_r)$ to expand Eq. (36) in a McLaurin series, we obtain

$$\Delta T_n \approx \Delta T_n(\tau\rightarrow\infty) - \frac{\tau_r}{\tau}\left[1+\exp\left(\frac{\tau_r}{\tau_R}\right)E_1\left(\frac{\tau_r}{\tau_R}\right)\right] \tag{56}$$

The derived small contribution to the solution Eq. (55) for ΔT_n $(\tau\rightarrow\infty)$ describes transient processes taking place in the system on a time scale significantly smaller than the modulation period τ.

From Eq. (55) the thermal-wave amplitudes increase without bound with increasing thermal resistance:

$$\Delta T_n(\tau_R \gg \tau_r) \approx \theta_r \ln(\tau_R/\tau_r)\rho_n \tag{57}$$

This occurs in the same manner as the contribution to the average field in the case of harmonic modulation. At this point it is important to recall that these solutions were derived in the quasi-stationary approximation, particularly under the condition $\tau_R \ll \tau$, and that they are limited (bounded) by the modulation period τ.

In the case of square-wave thermal-resistance modulation, it makes physical sense to examine one more (a third) limiting regime for the relative values of the characteristic times, that is $\tau_R \gg (\tau, 2\pi n\tau_r)$ in the system under consideration.

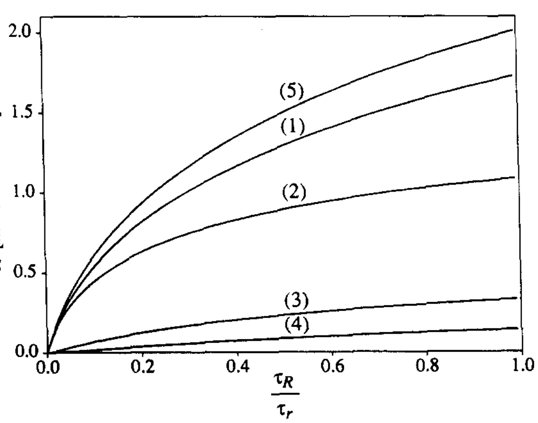

Fig. 5. Dependence of the dimensionless amplitudes of the spectral decomposition of the surface temperature, Eq. (50), on the parameter τ_R/τ_r in the quasi-stationary regime in the case of harmonic modulation of the thermal resistance. Curves (1)–(4), contributions to the average temperature A_0 and the amplitudes of the thermal-wave harmonics A_1–A_3; curve (5), $\Delta T_n/\theta_r\rho_n$ in the case of rectangular-wave periodic modulation of thermal resistance; from Eq. (55).

A simplification of Eq. (36) under this condition leads to:

$$\Delta T_n \approx \frac{\theta_r}{2\pi i n}\left\{\exp(i\pi n)\ln\left(1+\frac{\tau}{2\tau_r}\right) \times\exp\left(2\pi i n\frac{\tau_r}{\tau}\right) \times\left[E_1\left(-2\pi i n\frac{\tau_r}{\tau}-\pi i n\right) - E_1\left(-2\pi i n\frac{\tau_r}{\tau}\right)\right]\right\} \tag{58}$$

Eq. (58) shows that in the case of 100% modulation of the thermal resistance (this physically corresponds to the opening of a vacuum-filled closed crack) the amplitudes of all the thermal-wave spectral components are simply inversely proportional to H, since $\theta_r \approx 1/H$. The dependence of temperature variation on the parameter τ/τ_r, i.e. Eq. (58), describes the transition from 1-D to 3-D heat conduction with increasing value of τ/τ_r and contains no additional information about H. In the limit $\tau \gg 2\pi n\tau_r$, the solution Eq. (35) can be simplified

$$\Delta T_n \approx \theta_r \ln(\tau/\tau_r)\rho_n \tag{59}$$

thus demonstrating the logarithmic growth of temperature with the increase in modulation period τ under the conditions of 2-D heat conduction in the thin strip $0 \leqslant z \leqslant H$, which is thermally isolated from both sides.

The theory developed predicts that experimental observation of the onset of saturation of the thermal-wave amplitude with increasing parameter τ_R/τ_r, Fig. 5, leads to the identification of the situation when $\tau_R \approx \tau_r$. Thus the measurement of the quantity $HR_0 \approx r_0^2/4k$ is possible. In a manner similar to the 1-D geometry, the determination of R_0 may be achieved by the absolute measurements of temperature in the limit of weak modulation of the thermal resistance. This is possible because for $\tau_R/\tau_r \ll 1$ the amplitudes of the spectral components belonging both to $\Delta T(t)$ and $R(t)$ are proportional to $\theta_r(\tau_R/\tau_r) \approx J_L R_0$. The thickness H of the thin strip may also be extracted from the absolute measurements of temperatures in the limit of strong modulation of the thermal resistance when the amplitudes of all the thermal-wave spectral components are proportional to $1/H$, Eqs. (48) and (58).

It is interesting that in the case of weak harmonic modulation of the thermal resistance, $\tau_R \ll (\tau, \tau_r)$, both in the 1-D and in the quasi-stationary regimes the amplitudes of the higher harmonics $(n \geqslant 2)$ are still dependent on the overlayer thickness H:

$$\theta_{r,\omega_0,t} A_n \sim R_0{}^n H^{n-1} \tag{60}$$

Consequently, they are sensitive to the sub-surface position and the value of the thermal resistance. The measurement of these amplitudes should provide better contrast in the combined acousto-photo-thermal microscopy than in the conventional mode, which only detects the fundamental frequency.

Finally, let us describe the possible limits of the gas-filled layer thickness variation h in the above-developed theoretical model. The minimum values of h are determined by the condition (21) within the range of validity of the lumped heat-capacity analysis. Under the assumption $R \approx h/k_g$ this leads to the condition

$$h \gg \left(\frac{k_g}{k}\right) H \tag{61}$$

Note that for the gas–metal interface the typical situation is $(k_g/k) \sim 10^{-4}$. The maximum values of h are controlled by one of two factors. First, the physical concept of thermal resistance is valid only for thermally-thin gas layers: in 1-D, this requires the fulfilment of the inequality

$$h \ll \sqrt{D_g \tau} \tag{62}$$

D_g is the gas thermal diffusivity and $D_g/D \sim 1$ for a gas–metal combination. In the quasi-stationary regime the equivalent requirement to condition (62) is $h \ll r_0$. Second, in the 1-D regime ($\tau_r \gg \tau$, τ_R) the upper limit for h is set by the inequality $\tau_r \gg \tau_R$, which is equivalent to

$$h \ll \left(\frac{r_0}{H}\right)^2 \left(\frac{k_g}{k} H\right) \tag{63}$$

In the quasi-stationary regime ($\tau \gg \tau_r$, τ_R) the inequality $\tau \gg \tau_R$ yields an upper bound for h:

$$h \ll \left(\frac{\mathrm{D}\tau}{\mathrm{H}}\right)^2 \left(\frac{k_g}{k} H\right) \tag{64}$$

Therefore, under the condition (12) of a thermally-thin strip (overlayer), the combined inequalities shown below determine the possible non-zero range of h variations. For the 1-D regime these conditions are:

$$\frac{k_g}{k} H \ll h \ll \min\left[\sqrt{D_g \tau},\ \left(\frac{r_0}{H}\right)^2 \left(\frac{k_g}{k}\right)\right] \tag{65}$$

and for the quasi-stationary regime

$$\frac{k_g}{k} H \ll h \ll \min\left[r_0,\ \left(\frac{D\tau}{H}\right)^2 \left(\frac{k_g}{k} H\right)\right] \tag{66}$$

6. Conclusions

We investigated the process of parametric excitation of thermal waves by a sub-surface non-stationary defect localized in the path of a c.w. laser-induced heat flux. The analytical description of the thermal-wave frequency spectrum at the irradiated surface in the cases of harmonic and rectangular-wave periodic modulation of the thermal resistance of the defect by an external acoustic field has been presented. We described the role of multiple reflections of the thermal flux between the defect (crack) and the probed surface in synthesizing the amplitude and phase of the thermal-wave Fourier components at various frequencies. We also showed how these parametric processes cause broadening of the thermal-wave spectrum.

The proposed theory comprises a scheme of active acoustic modulation of the thermal resistance of the defect through its periodic opening by an applied stress field, which gives a methodology for characterizing sub-surface cracks which are photothermally invisible in the absence of acoustic (mechanical) loading. We also determined the conditions under which this combined acousto-photo-thermal materials diagnostic method can be applied to obtain information on the sub-surface localization of cracks.

This type of information is defect-selective in the sense that a signal appears at higher than fundamental harmonic frequencies only if a sub-surface defect exists, while the strong fundamental signal is efficiently suppressed by synchronous lock-in phase-sensitive means.

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References

1 J.-P. Monchalin, J.-L. Parpal, L. Bertrand and J.-M. Gagné, *Appl. Phys. Lett., 39* (1981) 391.
2 P.K. Kuo, L.D. Favro, L.J. Inglehart and R.L. Thomas, *J. Appl. Phys., 53* (1982) 1258.
3 G.C. Wetsel, Jr. and F.A. McDonald, *Appl. Phys. Lett., 41* (1982) 926.
4 P.M. Patel, D.P. Almond and H. Reiter, *Appl. Phys. B, 43* (1987) 9.
5 J.-P. Monchalin, J.-L. Parpal, L. Bertrand and J.-M. Gagné, *J. Appl. Phys., 53* (1982) 8525.
6 F.A. McDonald and G.C. Wetsel, in W.P. Mason and R.N. Thyrston (eds.), *Physical Acoustics, Vol. XVIII*, Academic Press, New York, 1988, p. 167.

7 E.H. Kennard, *Kinetic Theory of Gases*, McGraw-Hill, New York, 1983.
8 V.E. Gusev and A.A. Karabutov, *Laser Optoacoustics*, American Institute of Physics, New York, 1993.
9 P. Cielo, C.K. Jen and X. Maldague, *Can. J. Phys.*, *64* (1986) 1324.
10 G. Rousset, L. Bertrand and P. Cielo, *J. Appl. Phys.*, *57* (1985) 4396.
11 G.C. Wetsel, Jr. and J.B. Spicer, *Can. J. Phys.*, *64* (1986) 1269.
12 V. Gusev, A. Mandelis and R. Bleiss, *Appl. Phys. A*, *57* (1993) 229.
13 E. Castillo and M.R. Ruiz-Cobo, *Functional Equations and Modelling in Science and Engineering*, Marcel Dekker, New York, 1992.
14 W.G. Kelley and A.C. Peterson, *Difference Equations*, Academic Press, London, 1991.
15 V. Gusev, A. Mandelis and R. Bleiss, *Int. J. Thermophys.*, *14* (1993) 321.
16 J.P. Holman, *Heat Transfer*, McGraw-Hill, Englewood Cliffs, NJ, 5th edn., 1981.
17 M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions*, National Bureau of Standards, Washington, 1965.
18 A.P. Prudnikov, Yu.A. Brychkov and O.I. Marichev, *Integrals and Series, Vol. 1*, Gordon and Breach, Amsterdam, 1986.